# Steps Skipped in *Physical Mathematics*

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## Second-Edition List

## Chapter 13

### Section 13.18

In terms of Cartan's tetrads, the Christoffel symbol $\Gamma^k{}_{i\ell}$ is

$$
\begin{aligned}
\Gamma^k{}_{i\ell} &= \tfrac{1}{2}\, g^{kn}\left(g_{ni},_{\ell} + g_{n\ell},_{i} - g_{i\ell},_{n}\right) \\
&= \tfrac{1}{2} c^{ka} c^n{}_a \Big( \left(c_{nc} c_i{}^c\right)_{,\ell} + \left(c_{nc} c_\ell{}^c\right)_{,i} - \left(c_{ic} c_\ell{}^c\right)_{,n} \Big) \qquad (1) \\
&= \tfrac{1}{2} c^{ka} c^n{}_a \Big( c_{nc,\ell}\, c_i{}^c + c_{nc}\; c_i{}^c{}_{,\ell} + c_{nc,i}\, c_\ell{}^c + c_{nc}\; c_\ell{}^c{}_{,i} - c_{ic,n}\, c_\ell{}^c - c_{ic}\; c_\ell{}^c{}_{,n} \Big) \\
&= \tfrac{1}{2} c^{ka} c^n{}_a \Big( c_{nc,\ell}\, c_i{}^c + c_{nc,i}\, c_\ell{}^c - c_{ic,n}\, c_\ell{}^c - c_{ic}\; c_\ell{}^c{}_{,n} \Big) \\
&\quad + \tfrac{1}{2} c^{ka} c^n{}_a c_{nc} \big( c_i{}^c{}_{,\ell} + c_\ell{}^c{}_{,i} \big) \\
&= \tfrac{1}{2} c^{ka} c^n{}_a \Big( c_{nc,\ell}\, c_i{}^c + c_{nc,i}\, c_\ell{}^c - c_{ic,n}\, c_\ell{}^c - c_{ic}\; c_\ell{}^c{}_{,n} \Big) \\
&\quad + \tfrac{1}{2} c^{ka} \eta_{ac} \big( c_i{}^c{}_{,\ell} + c_\ell{}^c{}_{,i} \big) \\
&= \tfrac{1}{2} c^{ka} c^n{}_a \Big( c_{nc,\ell}\, c_i{}^c + c_{nc,i}\, c_\ell{}^c - c_{ic,n}\, c_\ell{}^c - c_{ic}\; c_\ell{}^c{}_{,n} \Big) + \tfrac{1}{2} c^{ka} \big( c_{ia,\ell} + c_{\ell a,i} \big).
\end{aligned}
$$

## Section 13.53

Thus using their orthonormality, we have $\omega^a{}_{d\,\ell}\, c^d{}_k\, c_b{}^k = \omega^a{}_{d\,\ell}\, \delta^d_b = \omega^a{}_{b\,\ell}$, and so the spin connection is

$$\begin{aligned}\omega^a{}_{b\,\ell} = &- c_b{}^k\big(c^a{}_{k,\ell} - \Gamma^j{}_{k\ell}\, c^a{}_j\big) = c^a{}_j\, c_b{}^k\, \Gamma^j{}_{k\ell} - c^a{}_{k,\ell}\, c_b{}^k \\ &= c^a{}_j\, c_b{}^k\, \Gamma^j{}_{k\ell} + c^a{}_k\, c_b{}^k{}_{,\ell}\end{aligned} \tag{2}$$

or

$$\omega^{ab}{}_i = c^a{}_j\, c^{bk}\, \Gamma^j{}_{ki} + c^a{}_k\, c^{bk}{}_{,i} \tag{3}$$

or

$$\omega^{ab}{}_i = c^a{}_j\, c^{bk}\, \Gamma^j{}_{ik} + c^a{}_k\, c^{bk}{}_{,i} \tag{4}$$

or

$$\omega^{ab}{}_i = c^a{}_k\, c^{b\ell}\, \Gamma^k{}_{i\ell} + c^a{}_k\, c^{bk}{}_{,i} \tag{5}$$

So

$$\begin{aligned}\omega^{ab}{}_i = {} & c^a{}_k\, c^{b\ell} \\ & \times \Big[\tfrac{1}{2} c^{kd} c^n{}_d \Big(c_{nc,\ell}\, c_i{}^c + c_{nc,i}\, c_\ell{}^c - c_{ic,n}\, c_\ell{}^c - c_{ic}\ c_\ell{}^c{}_{,n}\Big) + \tfrac{1}{2} c^{kd}\big(c_{id,\ell} + c_{\ell d,i}\big)\Big] \\ & + c^a{}_k\, c^{bk}{}_{,i} \\ = {} & \tfrac{1}{2}\, c^a{}_k\, c^{b\ell}\, c^{kd} c^n{}_d \Big(c_{nc,\ell}\, c_i{}^c + c_{nc,i}\, c_\ell{}^c - c_{ic,n}\, c_\ell{}^c - c_{ic}\ c_\ell{}^c{}_{,n}\Big) \\ & + \tfrac{1}{2}\, c^a{}_k\, c^{b\ell}\, c^{kd}\big(c_{id,\ell} + c_{\ell d,i}\big) + c^a{}_k\, c^{bk}{}_{,i} \\ = {} & \tfrac{1}{2}\, \eta^{ad}\, c^{b\ell}\, c^n{}_d \Big(c_{nc,\ell}\, c_i{}^c + c_{nc,i}\, c_\ell{}^c - c_{ic,n}\, c_\ell{}^c - c_{ic}\ c_\ell{}^c{}_{,n}\Big) \\ & + \tfrac{1}{2}\, \eta^{ad}\, c^{b\ell}\, \big(c_{id,\ell} + c_{\ell d,i}\big) + c^a{}_k\, c^{bk}{}_{,i} \\ = {} & \tfrac{1}{2}\ c^{b\ell}\, c^{an} \Big(c_{nc,\ell}\, c_i{}^c + c_{nc,i}\, c_\ell{}^c - c_{ic,n}\, c_\ell{}^c - c_{ic}\ c_\ell{}^c{}_{,n}\Big) \\ & + \tfrac{1}{2}\, c^{b\ell}\, \big(c^a{}_{i,\ell} + c^a{}_{\ell,i}\big) + c^a{}_k\, c^{bk}{}_{,i}.\end{aligned} \tag{6}$$

In more detail,

$$
\begin{aligned}
\omega^{ab}{}_i &= \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{nc,\ell}\, c_i{}^c + \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{nc,i}\, c_\ell{}^c \\
&\quad - \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{ic,n}\, c_\ell{}^c - \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{ic}\; c_\ell{}^c{}_{,n} \\
&\quad + \tfrac{1}{2}\, c^{b\ell}\left(c^a{}_{i,\ell} + c^a{}_{\ell,i}\right) + c^a{}_k\, c^{bk}{}_{,i} \\
&= \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{nc,\ell}\, c_i{}^c + \tfrac{1}{2}\,\eta^{bc}\, c^{an} c_{nc,i} \\
&\quad - \tfrac{1}{2}\,\eta^{bc}\, c^{an} c_{ic,n} - \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{ic}\; c_\ell{}^c{}_{,n} \\
&\quad + \tfrac{1}{2}\, c^{b\ell}\left(c^a{}_{i,\ell} + c^a{}_{\ell,i}\right) + c^a{}_k\, c^{bk}{}_{,i}.
\end{aligned} \tag{7}
$$

So using the $\eta$'s, we get

$$
\begin{aligned}
\omega^{ab}{}_i &= \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{nc,\ell}\, c_i{}^c + \tfrac{1}{2}\, c^{an}\, c^b{}_{n,i} \\
&\quad - \tfrac{1}{2}\, c^{an}\; c^b{}_{i,n} - \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{ic}\; c_\ell{}^c{}_{,n} \\
&\quad + \tfrac{1}{2}\, c^{b\ell}\left(c^a{}_{i,\ell} + c^a{}_{\ell,i}\right) + c^a{}_k\, c^{bk}{}_{,i}.
\end{aligned} \tag{8}
$$

Another way to write this is as

$$
\begin{aligned}
\omega^{ab}{}_i &= \tfrac{1}{2}\; c^{b\ell}\, c^{an} c_{cn,\ell}\, c^c{}_i + \tfrac{1}{2}\, c^{an}\left(c^b{}_{n,i} - c^b{}_{i,n}\right) \\
&\quad - \tfrac{1}{2}\; c^{b\ell}\, c^{an} c^c{}_i\; c_{c\ell,n} \\
&\quad + \tfrac{1}{2}\, c^{b\ell}\left(c^a{}_{i,\ell} + c^a{}_{\ell,i}\right) + c^a{}_k\, c^{bk}{}_{,i}
\end{aligned} \tag{9}
$$

or

$$
\begin{aligned}
\omega^{ab}{}_i &= \tfrac{1}{2}\, c^{aj}\left(c^b{}_{j,i} - c^b{}_{i,j}\right) + \tfrac{1}{2}\; c^{b\ell}\, c^{ak}\, c^c{}_i\left(c_{ck,\ell} - c_{c\ell,k}\right) \\
&\quad + \tfrac{1}{2}\, c^{b\ell}\left(c^a{}_{i,\ell} + c^a{}_{\ell,i}\right) + c^a{}_k\, c^{bk}{}_{,i}.
\end{aligned} \tag{10}
$$

This is the same as

$$
\begin{aligned}
\omega^{ab}{}_i &= \tfrac{1}{2}\, c^{aj}\left(c^b{}_{j,i} - c^b{}_{i,j}\right) + \tfrac{1}{2}\, c^{bj}\left(c^a{}_{i,j} + c^a{}_{j,i}\right) + c^a{}_j\, c^{bj}{}_{,i} \\
&\quad - \tfrac{1}{2}\, c^{ak}\, c^{b\ell}\; c^c{}_i\left(c_{c\ell,k} - c_{ck,\ell}\right)
\end{aligned} \tag{11}
$$

as well as

$$
\begin{aligned}
\omega^{ab}{}_i &= \tfrac{1}{2}\, c^{aj}\left(c^b{}_{j,i} - c^b{}_{i,j}\right) + \tfrac{1}{2}\, c^{bj}\left(c^a{}_{i,j} + c^a{}_{j,i}\right) - c^{bj} c^a{}_{j,i} \\
&\quad - \tfrac{1}{2}\, c^{ak}\, c^{b\ell}\; c^c{}_i\left(c_{c\ell,k} - c_{ck,\ell}\right)
\end{aligned} \tag{12}
$$

which is

$$\omega^{ab}{}_i = \tfrac{1}{2}\, c^{aj} \big( c^b{}_{j,i} - c^b{}_{i,j} \big) + \tfrac{1}{2}\, c^{bj} \, \big( c^a{}_{i,j} - c^a{}_{j,i} \big) \\ - \tfrac{1}{2}\, c^{ak}\, c^{b\ell}\; c^c{}_i \big( c_{c\ell,k} - c_{ck,\ell} \big) \tag{13}$$

or

$$\omega^{ab}{}_i = \tfrac{1}{2}\, c^{aj} \big( c^b{}_{j,i} - c^b{}_{i,j} \big) - \tfrac{1}{2}\, c^{bj} \, \big( c^a{}_{j,i} - c^a{}_{i,j} \big) \\ - \tfrac{1}{2}\, c^{ak}\, c^{b\ell}\; c^c{}_i \big( c_{c\ell,k} - c_{ck,\ell} \big) \tag{14}$$

or (Deser and Isham, 1976; Green et al., 1987)

$$\omega^{ab}{}_i = \tfrac{1}{2} c^{aj} \left( \partial_i c^b{}_j - \partial_j c^b{}_i \right) - \tfrac{1}{2} c^{bj} \left( \partial_i c^a{}_j - \partial_j c^a{}_i \right) \\ - \tfrac{1}{2} c^{ak} c^{b\ell} c^c{}_i \left( \partial_k c_{c\ell} - \partial_\ell c_{ck} \right). \tag{15}$$

Incidentally, the last two equations imply that the affine connection $\Gamma^i{}_{k\ell}$ is

$$\Gamma^i{}_{k\ell} = c_a{}^i \Big[ \partial_\ell c_a{}^k + \tfrac{1}{2} c_b{}^k \Big( c^{aj} (\partial_\ell c^b{}_j - \partial_j c^b{}_\ell) - c^{bj} (\partial_\ell c^a{}_j - \partial_j c^a{}_\ell) \\ - c^{ak} c^{bm} c^c{}_\ell (\partial_k c_{cm} - \partial_\ell c_{ck}) \Big) \Big]. \tag{16}$$