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Nuclear Physics B (Proc. Suppl.) 42 (1995) 858–860

NUCLEAR PHYSICS B
PROCEEDINGS
SUPPLEMENTS

Noncompact Gauge-Invariant Simulations of $U(1)$, $SU(2)$, and $SU(3)$

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We have applied a new gauge-invariant, noncompact, Monte Carlo method to simulate $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on 8^4 and 12^4 lattices. The Creutz ratios of the Wilson loops agree with the exact results for $U(1)$ for $\beta \geq 1$ apart from a renormalization of the charge. The $SU(2)$ and $SU(3)$ Creutz ratios robustly display quark confinement at $\beta = 0.5$ and $\beta = 2$, respectively. At much weaker coupling, the $SU(2)$ and $SU(3)$ Creutz ratios agree with perturbation theory after a renormalization of the coupling constant. For $SU(3)$ the scaling window is near $\beta = 2$, and the relation between the string tension σ and our lattice QCD parameter Λ_L is $\sqrt{\sigma} \approx 5\Lambda_L$.

1. INTRODUCTION

In compact lattice gauge theory, gauge fields are represented by group elements rather than by fields, and the action is a *periodic* function of a gauge-invariant lattice field strength. The periodicity of the action entails spurious vacua. The principal advantage of noncompact actions, in which gauge fields are represented by fields, is that they avoid multiple vacua.

The first gauge-invariant noncompact simulations were carried out by Palumbo, Polikarpov, and Veselov [1]. They saw a confinement signal. Their action contains five terms, constructed from two invariants, and involves (noncompact) auxiliary fields and an adjustable parameter.

The present paper describes a test of a new way [2] of performing gauge-invariant noncompact simulations. Our action, which is similar to one term of Palumbo's action, is exactly invariant under compact gauge transformations, is a natural discretization of the classical Yang-Mills action, and reduces to Wilson's action when the gauge fields are compactified. In this method there are fewer auxiliary fields than in Palumbo's method, and they are compact group elements representing gauge transformations.

We have used this method to simulate $U(1)$,

$SU(2)$, and $SU(3)$ gauge theories on 8^4 and 12^4 lattices. The Creutz ratios of Wilson loops agree with the exact results for $U(1)$ for $\beta \geq 1$ apart from a renormalization of the charge. The $SU(2)$ and $SU(3)$ Creutz ratios clearly show quark confinement at $\beta = 0.5$ and $\beta = 2$, respectively. At much weaker coupling, the $SU(2)$ and $SU(3)$ Creutz ratios agree with perturbation theory with a renormalized coupling constant. For $SU(3)$ there is a scaling window near $\beta = 2$, and the string tension σ is related to the lattice QCD parameter Λ_L by $\sqrt{\sigma} \approx 5\Lambda_L$. If $\sqrt{\sigma} \approx 420$ MeV, then our Λ_L is about 84 MeV, and at $\beta = 2$ our lattice spacing a is about 0.4 fm.

2. THE METHOD

For massless fermions, the continuum action density is $\bar{\psi}i\gamma_\mu\partial_\mu\psi$. A suitable discretization of this quantity is $i\bar{\psi}(n)\gamma_\mu[\psi(n+e_\mu) - \psi(n)]/a$ in which n is a four-vector of integers representing an arbitrary vertex of the lattice, e_μ is a unit vector in the μ th direction, and a is the lattice spacing. The product of Fermi fields at the same point is gauge invariant as it stands. The other product of Fermi fields becomes gauge invariant if we insert a matrix $A_\mu(n)$ of gauge fields

$$\bar{\psi}(n)\gamma_\mu [1 + igaA_\mu(n)] \psi(n + e_\mu) \quad (1)$$

that transforms appropriately. Under a gauge transformation represented by the group elements

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$U(n)$ and $U(n + e_\mu)$, the required response is

$$1 + iagA'_\mu(n) = U(n)[1 + iagA_\mu(n)]U^{-1}(n + e_\mu). \quad (2)$$

Under this gauge transformation, the lattice field strength

$$\begin{aligned} F_{\mu\nu}(n) = & \frac{1}{a} [A_\mu(n + e_\nu) - A_\mu(n)] \\ & - \frac{1}{a} [A_\nu(n + e_\mu) - A_\nu(n)] \\ & + ig [A_\nu(n)A_\mu(n + e_\nu) \\ & - A_\mu(n)A_\nu(n + e_\mu)], \end{aligned} \quad (3)$$

which reduces to the continuum Yang-Mills field strength in the limit $a \rightarrow 0$, transforms as

$$F'_{\mu\nu}(n) = U(n)F_{\mu\nu}(n)U^{-1}(n + e_\mu + e_\nu). \quad (4)$$

The field strength $F_{\mu\nu}(n)$ is antisymmetric in the indices μ and ν , but it is not hermitian. To make a positive plaquette action density, we use the Hilbert-Schmidt norm of $F_{\mu\nu}(n)$

$$S = \frac{1}{4k} \text{Tr}[F_{\mu\nu}^\dagger(n)F_{\mu\nu}(n)], \quad (5)$$

in which the generators T_a of the gauge group are normalized as $\text{Tr}(T_a T_b) = k\delta_{ab}$. Because $F_{\mu\nu}(n)$ transforms covariantly (4), this action density is exactly invariant under the noncompact gauge transformation (2).

In general the gauge transformation (2) with group element $U(n) = \exp(-iag\omega^a T_a)$ maps the matrix of gauge fields $A_\mu(n) = T_a A_\mu^a(n)$ outside the Lie algebra, apart from terms of lowest (zeroth) order in the lattice spacing a . We use this larger space of matrices. We use the action (5) in which the field strength (3) is defined in terms of gauge-field matrices $A_\mu(n)$ that are the images under arbitrary gauge transformations

$$A_\mu(n) = V A_\mu^0(n) W^{-1} + \frac{i}{ag} V (V^{-1} - W^{-1}) \quad (6)$$

of matrices $A_\mu^0(n)$ of gauge fields defined in the usual way, $A_\mu^0(n) \equiv T_a A_\mu^{a,0}(n)$. The group elements V and W associated with the gauge field $A_\mu(n)$ are unrelated to those associated with the neighboring gauge fields $A_\mu(n + e_\nu)$, $A_\nu(n)$, and $A_\nu(n + e_\mu)$.

The quantity $1 + iagA_\mu(n)$ is not an element $L_\mu(n)$ of the gauge group. But if one compactified the fields by requiring $1 + iagA_\mu(n)$ to be an element of the gauge group, then the matrix $A_\mu(n)$ of gauge fields would be related to the link $L_\mu(n)$ by $A_\mu(n) = (L_\mu(n) - 1)/(iga)$, and the action (5) defined in terms of the field strength (3) would be, *mirabile dictu*, Wilson's action:

$$S = \frac{k - \Re \text{Tr} L_\mu(n) L_\nu(n + e_\mu) L_\mu^\dagger(n + e_\nu) L_\nu^\dagger(n)}{2a^4 g^2 k}.$$

3. RESULTS

We have tested this method by applying it to the $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on 8^4 and 12^4 lattices. In most of our initial configurations, the unitary matrices V and W and the hermitian gauge fields A_μ^0 were randomized. For thermalization we allowed 50,000 sweeps for $U(1)$, 10,000 for $SU(2)$, and 100,000 for $SU(3)$. Our Wilson loops are ensemble averages of ordered products of the binomials $1 + iagA_\mu(n)$ rather than of the exponentials $\exp[iagA_\mu(n)]$ around the loop.

For $U(1)$ and for $\beta \geq 1$, our measured Creutz ratios [3] of Wilson loops agree with the exact ones apart from finite-size effects and a renormalization of the charge. For instance at $\beta = 1$ on the 12^4 lattice, we found $\chi(2, 2) = 0.147(1)$, $\chi(2, 3) = 0.103(1)$, $\chi(2, 4) = 0.090(1)$, $\chi(3, 3) = 0.049(1)$, $\chi(3, 4) = 0.034(1)$, and $\chi(4, 4) = 0.020(2)$. The first three of these χ 's are equal to the exact Creutz ratios for a renormalized value of $\beta_r = 0.9$; the last three are smaller than the exact ratios for $\beta_r = 0.9$ due to finite-size effects by 3%, 8%, and 16%, respectively.

But at stronger coupling, the extra terms $ig [A_\nu(n)A_\mu(n + e_\nu) - A_\mu(n)A_\nu(n + e_\mu)]$ in the lattice field strength $F_{\mu\nu}(n)$ eventually do produce a confinement signal. For example, at $\beta = 0.75$, our measured Creutz ratios on the 12^4 lattice are: $\chi(2, 2) = 0.906(5)$, $\chi(2, 3) = 0.909(21)$, $\chi(2, 4) = 0.85(10)$, $\chi(3, 3) = 0.62(24)$, and $\chi(3, 4) = 0.6(16)$.

For $SU(2)$ on the 8^4 lattice at $\beta = 0.5$, we found $\chi(2, 2) = 0.835(3)$, $\chi(2, 3) = 0.852(12)$, $\chi(2, 4) = 0.865(60)$, and $\chi(3, 3) = 0.94(23)$ which within the limited statistics clearly exhibit con-

finement. At $\beta = 1$, our six Creutz ratios track those of tree-level perturbation theory for a renormalized value of $\beta_r = 1.75$.

For $SU(3)$ at $\beta = 2$ on the 12^4 lattice, we found in ten independent runs $\chi(2, 2) = 0.838(1)$, $\chi(2, 3) = 0.826(3)$, $\chi(2, 4) = 0.828(13)$, $\chi(3, 3) = 0.793(42)$, $\chi(3, 4) = 0.47(25)$, and $\chi(4, 4) = 1.2(86)$. Within the statistics, these results robustly exhibit confinement. At much weaker coupling, our ratios agree with perturbation theory apart from finite-size effects and after a renormalization of the coupling constant.

4. SCALING

We used an 8^4 lattice to study the scaling of the lattice spacing a with the coupling constant g for $SU(3)$. The two-loop result for the dependence of the string tension σa^2 upon the inverse coupling $\beta = 6/g^2$ is

$$\sigma a^2 \approx \frac{\sigma}{\Lambda_L^2} \exp \left[-\frac{8\pi^2\beta}{33} + \frac{102}{121} \log \left(\frac{8\pi^2\beta}{33} \right) \right]. \quad (7)$$

If we set $\sqrt{\sigma} \approx (5.0 \pm 0.4)\Lambda_L$, then our $\chi(i, j)$'s fit this formula for $1.9 < \beta < 2.1$ as shown in the figure. A string tension $\sqrt{\sigma} \approx 420$ MeV implies that $\Lambda_L \approx 84$ MeV, which is about 11 times closer to the continuum $\Lambda_{\overline{MS}}^{(0)}$ than is the parameter $\Lambda_{LW} \approx 7.9$ MeV of Wilson's method. At $\beta = 2$, our lattice spacing a is about 0.4 fm.

ACKNOWLEDGMENTS

We are indebted to M. Creutz, G. Marsaglia, F. Palumbo, W. Press, and K. Webb for useful conversations, to the Department of Energy for financial support under task B of grant DE-FG03-92ER40732/A004, and to B. Dieterle and the Maui Center for High-Performance Computing¹ for computer time.

¹Research sponsored in part by the Phillips Laboratory, Air Force Materiel Command, USAF, under cooperative agreement F29601-93-2-0001. The U.S. Government retains a nonexclusive copyright to this work. The views and conclusions of this work are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of Phillips Laboratory or the U.S. Government.

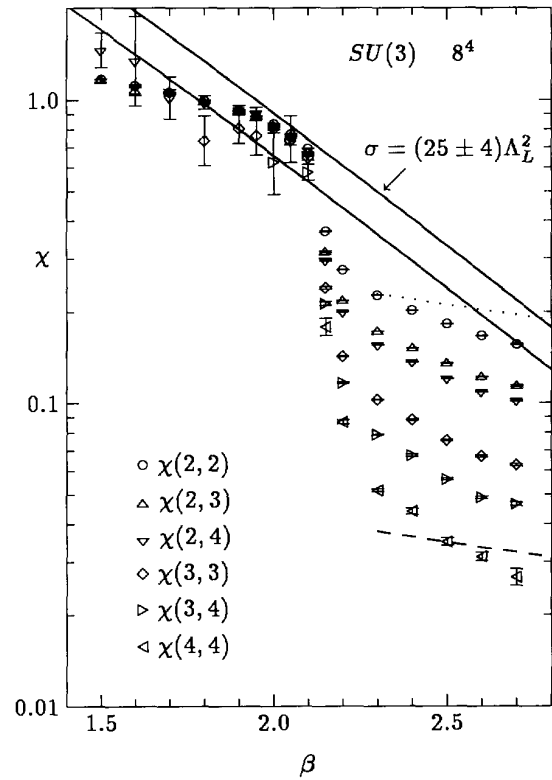


Figure 1. The $SU(3)$ Creutz ratios $\chi(i, j)$, the scaling predictions for the string tension σa^2 , and the tree-level curves for $\chi(2, 2)$ (dots) and $\chi(4, 4)$ (dashes) are plotted against β . For $1.9 < \beta < 2.1$, some of the symbols of the χ 's overlap.

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