

Path integrals for actions that are not quadratic in their time derivatives

Kevin Cahill*

Department of Physics & Astronomy, University of New Mexico,

Albuquerque, New Mexico 87131, USA and

School of Computational Sciences,

Korea Institute for Advanced Study,

Seoul 130-722, Korea

(Dated: April 17, 2015)

Abstract

The standard way to construct a path integral is to use a Legendre transformation to find the hamiltonian, to repeatedly insert complete sets of states into the time-evolution operator, and then to integrate over the momenta. This procedure is simple when the action is quadratic in its time derivatives, but in most other cases Legendre's transformation is intractable, and the hamiltonian is unknown. This paper shows how to construct path integrals when one can't find the hamiltonian because the first time derivatives of the fields occur in ways that make a Legendre transformation intractable; it focuses on scalar fields and does not discuss higher-derivative theories or those in which some fields lack time derivatives.

* kevinecahill@gmail.com

I. INTRODUCTION

Despite the success of renormalization, infinities remain a major problem in quantum field theory, one that has grown more acute as cosmological observations have confirmed the reality of dark energy [1]. For if dark energy is the energy of the vacuum, then we need to be able to compute energies in finite theories. The ground-state energy of a theory with hamiltonian H is the big β limit of $-d \ln \text{Tr} [\exp(-\beta H)] / d\beta$. We can study the ground-state energy of a theory if we can write the partition function $Z(\beta) = \text{Tr} [\exp(-\beta H)]$ as a path integral in euclidian space.

If the action is quadratic in the first time derivatives of the fields, then the hamiltonian is a simple Legendre transform of the Lagrange density, and we can use it to construct the path integral in the standard manner [2]. But if the action is not quadratic in the time derivatives of the fields, then the Legendre transform may be impossible, the hamiltonian unknown, and the path integral a mystery.

Some quantum field theories have finite Green's functions and finite euclidian action densities [3, 4], but they have actions that are not quadratic in the time derivatives of the fields. The Nambu-Gotō action of string theory is not quadratic in the τ derivatives of the fields $X^\mu(\sigma, \tau)$. Apart from theories with unbroken supersymmetry, theories with finite energy densities tend to have actions that are not quadratic in the time derivatives of the fields. The hamiltonians and path integrals of these theories are either complicated or unknown. This paper shows how to construct path integrals when one can't find the hamiltonian because the first time derivatives of the fields occur in ways that make a Legendre transformation intractable; it focuses on scalar fields and does not discuss higher-derivative theories [5] or those in which some fields lack time derivatives [6].

II. LAGRANGIANS AND HAMILTONIANS

The lagrangian of a theory tells us about symmetries and equations of motion, and the hamiltonian tells us how to construct path integrals and how to compute the time evolution of states and their energies. The standard way to construct a path integral is to use a Legendre transformation to find the hamiltonian H from its lagrangian L and to insert complete sets of eigenstates of the fields ϕ_j and of their conjugate momenta π_j into the

time-evolution operator $\exp(-itH)$ so as to get

$$\langle \phi'' | e^{-itH} | \phi' \rangle = \int \exp \left\{ \int i \left[\dot{\phi}_j \pi_j - H(\phi, \pi) \right] d^4x \right\} D\phi D\pi \quad (1)$$

in which the time integral is from 0 to t , and the letters ϕ and π stand for all the fields ϕ_1, \dots, ϕ_n and momenta π_1, \dots, π_n of the action [2]. If one can integrate over the momenta and does so, then one has the usual expression

$$\langle \phi'' | e^{-itH} | \phi' \rangle = \int \exp \left[\int i L(\phi, \dot{\phi}) d^4x \right] D\phi. \quad (2)$$

This procedure is straightforward when the lagrangian is quadratic in its time derivatives, but in most other theories the formulas that express the time derivatives in terms of the fields and their momenta are insoluble and the hamiltonian is unknown. Most theories are therefore inaccessible and unexamined.

The solution to this problem is to let functional integration perform Legendre's transformation (Sec. III) implicitly. Delta functionals can impose the relation between the time derivatives and the fields and their momenta as illustrated by four examples in Sec. IV. The cost is a doubling of the fields over which one must integrate and a determinant that makes a ghostly appearance when it is positive (Sec. V). Similar delta functionals work in euclidian space (Sec. VI). The Nambu-Goto action of string theory is not quadratic in its time derivatives; its path integral is exhibited in Sec. VII.

III. THE LEGENDRE TRANSFORMATION

In theories of scalar fields, the momenta are derivatives of the action density

$$\pi_j = \frac{\partial L}{\partial \dot{\phi}_j}. \quad (3)$$

If one can invert these equations and write the time derivatives $\dot{\phi}_j$ of the fields in terms of the fields ϕ_ℓ and their momenta π_ℓ , then the energy density is

$$H = \sum_{j=1}^n \pi_j \dot{\phi}_j(\phi, \pi) - L(\phi, \dot{\phi}(\phi, \pi)). \quad (4)$$

When the action is quadratic in all the time derivatives, this Legendre transformation is easy to do, but in most other cases no solution is known even in the absence of constraints.

IV. A PATH-INTEGRAL LEGENDRE TRANSFORMATION

Let the field $\dot{\chi}(\phi, \pi)$ be the function of the fields ϕ_j and their conjugate momenta π_j that satisfies Legendre's relation

$$\pi_j = \frac{\partial L(\phi, \dot{\chi})}{\partial \dot{\chi}_j}. \quad (5)$$

In terms of this in-general-inaccessible function $\dot{\chi}(\phi, \pi)$, the energy density is

$$H(\phi, \pi) = \sum_{j=1}^n \pi_j \dot{\chi}_j(\phi, \pi) - L(\phi, \dot{\chi}(\phi, \pi)) \quad (6)$$

and the path integral (1) is

$$\langle \phi'' | e^{-itH} | \phi' \rangle = \int \exp \left\{ \int i \left[\dot{\phi}_j \pi_j - (\pi_k \dot{\chi}_k(\phi, \pi) - L(\phi, \dot{\chi}(\phi, \pi))) \right] d^4x \right\} D\phi D\pi. \quad (7)$$

Although we don't know what $\dot{\chi}(\phi, \pi)$ is, we still can write this path integral in terms of a delta functional as

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[\dot{\phi}_j \pi_j - (\pi_k \dot{\chi}_k(\phi, \pi) - L(\phi, \dot{\chi}(\phi, \pi))) \right] d^4x \right\} \\ &\times \prod_{\ell, x} \left[\delta \left(\dot{\chi}_\ell(\phi, \pi) - \dot{\psi}_\ell \right) \right] D\phi D\pi D\dot{\psi}. \end{aligned} \quad (8)$$

We can use the delta functional to replace $\dot{\chi}$ by $\dot{\psi}$ everywhere else:

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[\dot{\phi}_j \pi_j - (\pi_k \dot{\psi}_k(\phi, \pi) - L(\phi, \dot{\psi}(\phi, \pi))) \right] d^4x \right\} \\ &\times \prod_{\ell, x} \left[\delta \left(\dot{\chi}_\ell(\phi, \pi) - \dot{\psi}_\ell \right) \right] D\phi D\pi D\dot{\psi}. \end{aligned} \quad (9)$$

The delta functional $\delta(\dot{\chi} - \dot{\psi})$ imposes the relation (5) among the fields ϕ , π , and $\dot{\psi}$. We now use the delta-function identity

$$\begin{aligned} \prod_{\ell, x} \left[\delta \left(\dot{\chi}_\ell(\phi, \pi) - \dot{\psi}_\ell \right) \right] &= \prod_{\ell, x} \left[\delta \left(\pi_\ell - \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} \right) \right] \left| \det \left(\frac{\partial \pi_k}{\partial \dot{\psi}_\ell} \right) \right| \\ &= \prod_{\ell, x} \left[\delta \left(\pi_\ell - \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} \right) \right] \left| \det \left(\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| \end{aligned} \quad (10)$$

to change variables in the delta functional introducing the appropriate jacobian

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[\dot{\phi}_j \pi_j - (\pi_k \dot{\psi}_k - L(\phi, \dot{\psi})) \right] d^4x \right\} \\ &\times \prod_{\ell, x} \left[\delta \left(\pi_\ell - \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} \right) \right] \left| \det \left(\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| D\phi D\pi D\dot{\psi}. \end{aligned} \quad (11)$$

The repeated indices j, k, ℓ are summed from 1 to n . The integration is over all fields that go from ϕ' to ϕ'' in time t and over all π and $\dot{\psi}$ in that time interval. Integrating over π , we find

$$\langle \phi'' | e^{-itH} | \phi' \rangle = \int \exp \left[\int i (\dot{\phi}_\ell - \dot{\psi}_\ell) \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} + i L(\phi, \dot{\psi}) d^4x \right] \left| \det \left(\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| D\phi D\dot{\psi}. \quad (12)$$

This functional integral generalizes the path integral to theories of scalar fields in which the action is not quadratic in the time derivatives of the fields. A similar formula should work in theories of vector and tensor fields, apart from the issue of constraints.

Our first example is a free scalar field with action density

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (13)$$

and canonical momentum

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}. \quad (14)$$

The determinant is unity, and the proposed path integral (12) is

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[L(\phi, \dot{\psi}) + \dot{\psi}(\dot{\phi} - \dot{\psi}) \right] d^4x \right\} D\phi D\dot{\psi} \\ &= \int \exp \left\{ \int i \left[\frac{1}{2} \dot{\psi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 + \dot{\psi}(\dot{\phi} - \dot{\psi}) \right] d^4x \right\} D\phi D\dot{\psi}. \end{aligned} \quad (15)$$

Shifting $\dot{\psi}$ to $\dot{\psi} + \dot{\phi}$ and integrating over $\dot{\psi}$, we get the usual result

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[\frac{1}{2} (\dot{\psi} + \dot{\phi})^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 - (\dot{\psi} + \dot{\phi}) \dot{\psi} \right] d^4x \right\} D\phi D\dot{\psi} \\ &= \int \exp \left\{ \int i \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \dot{\psi}^2 \right] d^4x \right\} D\phi D\dot{\psi} \\ &= \int \exp \left\{ \int i \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 \right] d^4x \right\} D\phi \\ &= \int \exp \left[i \int L(\phi, \dot{\phi}) d^4x \right] D\phi. \end{aligned} \quad (16)$$

Our second example is a field theory in one dimension, time. The lagrangian for a relativistic particle of mass m

$$L = -m \sqrt{1 - \dot{q}^2} \quad (17)$$

is not quadratic the first time derivative \dot{q} . This theory is special in that we can invert the definition of the momentum

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{m\dot{q}}{\sqrt{1 - \dot{q}^2}}, \quad (18)$$

express \dot{q} in terms of p

$$\dot{q} = \frac{p}{\sqrt{p^2 + m^2}}, \quad (19)$$

and find the hamiltonian

$$H = \sqrt{p^2 + m^2}. \quad (20)$$

The standard double path integral (1) then is

$$\langle q'' | e^{-itH} | q' \rangle = \int \exp \left[\int i \left(\dot{q}p - \sqrt{p^2 + m^2} \right) dt \right] Dq Dp. \quad (21)$$

Suppose, however, that we were unable to perform Legendre's transformation and get these equations (19–21). In that case, we could use the proposed path integral (12)

$$\langle q'' | e^{-itH} | q' \rangle = \int \exp \left[\int i \left(\dot{q} - \dot{s} \right) \frac{\partial L(q, \dot{s})}{\partial \dot{s}} + iL(q, \dot{s}) \right] \left| \frac{\partial^2 L(q, \dot{s})}{\partial \dot{s}^2} \right| Dq D\dot{s}. \quad (22)$$

We then would write

$$\langle q'' | e^{-itH} | q' \rangle = \int \exp \left[\int i \left(\dot{q} - \dot{s} \right) \frac{m\dot{s}}{\sqrt{1 - \dot{s}^2}} - im\sqrt{1 - \dot{s}^2} \right] \frac{m}{(1 - \dot{s}^2)^{3/2}} Dq D\dot{s}. \quad (23)$$

To check that this path integral (23) is the same as the standard double path integral (21), we change variables in it (23), setting

$$\dot{s} = \frac{p}{\sqrt{p^2 + m^2}} \quad (24)$$

so that

$$d\dot{s} = \frac{m^2}{(p^2 + m^2)^{3/2}} dp. \quad (25)$$

We then find that

$$\frac{m}{(1 - \dot{s}^2)^{3/2}} d\dot{s} = m \left(\frac{p^2 + m^2}{m^2} \right)^{3/2} \frac{m^2}{(p^2 + m^2)^{3/2}} dp = dp, \quad (26)$$

and that

$$\begin{aligned} (\dot{q} - \dot{s}) \frac{m\dot{s}}{\sqrt{1 - \dot{s}^2}} - m\sqrt{1 - \dot{s}^2} &= \dot{q}p - \frac{p^2}{\sqrt{p^2 + m^2}} - \frac{m^2}{\sqrt{p^2 + m^2}} \\ &= \dot{q}p - \sqrt{p^2 + m^2}. \end{aligned} \quad (27)$$

Thus the proposed path integral (12) reduces in this quantum-mechanical example to a path integral (22) that is the same as the standard double path integral (21) for this example.

Our third example is the scalar Born-Infeld theory with action density

$$L = -M^4 \sqrt{1 - M^{-4} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right)} \quad (28)$$

which is the field-theory version of the second example. The proposed path integral (12) is

$$\langle \phi'' | e^{-itH} | \phi' \rangle = \int \exp \left\{ \int i \left[(\dot{\phi} - \dot{\psi}) \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} + L(\phi, \dot{\psi}) \right] d^4x \right\} \left| \frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2} \right| D\phi D\dot{\psi} \quad (29)$$

in which the partial derivatives are

$$\pi(\phi, \dot{\psi}) = \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} = \frac{\dot{\psi}}{\sqrt{1 - M^{-4} \left(\dot{\psi}^2 - (\nabla\psi)^2 - m^2\phi^2 \right)}} \quad (30)$$

and

$$\frac{\partial \pi(\phi, \dot{\psi})}{\partial \dot{\psi}} = \frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2} = \frac{1 + M^{-4} \left((\nabla\psi)^2 + m^2\phi^2 \right)}{\sqrt{1 - M^{-4} \left(\dot{\psi}^2 - (\nabla\psi)^2 - m^2\phi^2 \right)}}. \quad (31)$$

Substituting these formulas into the proposed path integral (29) gives

$$\langle \phi'' | e^{-itH} | \phi' \rangle = \int \exp \left\{ \int i \left[(\dot{\phi} - \dot{\psi}) \pi(\phi, \dot{\psi}) + L(\phi, \dot{\psi}) \right] d^4x \right\} \frac{\partial \pi(\phi, \dot{\psi})}{\partial \dot{\psi}} D\phi D\dot{\psi}. \quad (32)$$

This theory is one of the few in which we can solve (30) for the time derivative $\dot{\psi}$

$$\dot{\psi} = \frac{\pi}{\sqrt{1 + M^{-4} \pi^2}} \sqrt{1 + M^{-4} \left((\nabla\phi)^2 + m^2\phi^2 \right)} \quad (33)$$

and find as the hamiltonian density

$$\begin{aligned} H(\phi, \pi(\phi, \dot{\psi})) &= \pi(\phi, \dot{\psi}) \dot{\psi} - L(\phi, \dot{\psi}) \\ &= \sqrt{(M^4 + \pi^2) (M^4 + (\nabla\phi)^2 + m^2\phi^2)}. \end{aligned} \quad (34)$$

Thus the proposed path integral (32) is the standard formula (1)

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[\dot{\phi} \pi(\phi, \dot{\psi}) - H(\phi, \pi(\phi, \dot{\psi})) \right] d^4x \right\} D\phi D\dot{\psi} \frac{\partial \pi}{\partial \dot{\psi}} \\ &= \int \exp \left\{ \int i \left[\dot{\phi} \pi - H(\phi, \pi) \right] d^4x \right\} D\phi D\pi \\ &= \int \exp \left\{ \int i \left[\dot{\phi} \pi - \sqrt{(M^4 + \pi^2) (M^4 + (\nabla\phi)^2 + m^2\phi^2)} \right] d^4x \right\} D\phi D\pi. \end{aligned} \quad (35)$$

Our fourth example is the theory defined by the action density

$$L = M^4 \exp(L_0/M^4) \quad (36)$$

in which L_0 is the action density (13) of the free field. The derivatives of L are

$$\frac{\partial L}{\partial \dot{\psi}} = M^{-4} \dot{\psi} L \quad \text{and} \quad \frac{\partial^2 L}{\partial \dot{\psi}^2} = M^{-4} (1 + M^{-4} \dot{\psi}^2) L. \quad (37)$$

So the proposed path integral is

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[L(\phi, \dot{\psi}) + \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} (\dot{\phi} - \dot{\psi}) \right] d^4x \right\} \left| \frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2} \right| D\phi D\dot{\psi} \\ &= \int \exp \left\{ \int i \left[1 + \frac{\dot{\psi}(\dot{\phi} - \dot{\psi})}{M^4} \right] L(\phi, \dot{\psi}) d^4x \right\} \frac{(1 + M^{-4} \dot{\psi}^2) L(\phi, \dot{\psi})}{M^4} D\phi D\dot{\psi}. \end{aligned} \quad (38)$$

V. HIDDEN FERMIONIC VARIABLES

A determinant is a gaussian integral

$$\int e^{-\theta^\dagger A \theta} \prod_{k=1}^n d\theta_k^* d\theta_k = \det A \quad (39)$$

as is well-known [2]. When the determinant is positive, we can drop the absolute-value signs and write the proposed path integral (12) as

$$\begin{aligned} \langle \phi'' | e^{-itH} | \phi' \rangle &= \int \exp \left\{ \int i \left[(\dot{\phi}_\ell - \dot{\psi}_\ell) \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} + L(\phi, \dot{\psi}) \right] d^4x \right\} \det \left(\frac{\partial^2 L}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) D\phi D\dot{\psi} \\ &= \int \exp \left\{ \int i (\dot{\phi}_\ell - \dot{\psi}_\ell) \frac{\partial L}{\partial \dot{\psi}_\ell} + i L - \bar{\chi}_k \frac{\partial^2 L}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \chi_\ell d^4x \right\} D\phi D\dot{\psi} D\bar{\chi} D\chi \end{aligned} \quad (40)$$

in which ϕ_ℓ and ψ_ℓ are boson fields, and χ_k are scalar Grassmann fields or ghosts.

VI. EUCLIDIAN SPACE

It is easier to evaluate path integrals in euclidian space where in the proposed path integral (12) the integral over x^4 runs from 0 to the inverse temperature $\beta = 1/kT$

$$\langle \phi'' | e^{-\beta H} | \phi' \rangle = \int \exp \left[\int (i \dot{\phi}_j - \dot{\psi}_j) \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_j} + L(\phi, \dot{\psi}) d^4x \right] \left| \det \left(\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| D\phi D\dot{\psi} \quad (41)$$

in which L is the action density in Minkowski space.

In this theory, the mean value of an observable $A[\phi]$ in a system at maximum entropy and inverse temperature β is

$$\begin{aligned} \langle A[\phi] \rangle &= \frac{\text{Tr } A[\phi] e^{-\beta H}}{\text{Tr } e^{-\beta H}} = \int A[\phi] \exp \left[\int \left(i \dot{\phi}_j - \dot{\psi}_j \right) \frac{\partial L}{\partial \dot{\psi}_j} + L d^4 x \right] \left| \det \left(\frac{\partial^2 L}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| D\phi D\dot{\psi} \\ &\quad / \int \exp \left[\int \left(i \dot{\phi}_j - \dot{\psi}_j \right) \frac{\partial L}{\partial \dot{\psi}_j} + L d^4 x \right] \left| \det \left(\frac{\partial^2 L}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| D\phi D\dot{\psi}. \end{aligned} \quad (42)$$

This ratio of path integrals is a ratio of mean values

$$\langle A[\phi] \rangle = \left\langle A[\phi] \exp \left[\int i \dot{\phi}_j \frac{\partial L}{\partial \dot{\psi}_j} d^4 x \right] \right\rangle / \left\langle \exp \left[\int i \dot{\phi}_j \frac{\partial L}{\partial \dot{\psi}_j} d^4 x \right] \right\rangle \quad (43)$$

each estimated by Monte Carlo simulation [7] in the normalized probability distribution

$$\begin{aligned} P(\phi, \dot{\psi}) &= \exp \left[\int \left(L(\phi, \dot{\psi}) - \dot{\psi}_j \frac{\partial L}{\partial \dot{\psi}_j} \right) d^4 x \right] \left| \det \left(\frac{\partial^2 L}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| \\ &\quad / \int \exp \left[\int \left(L(\phi, \dot{\psi}) - \dot{\psi}_j \frac{\partial L}{\partial \dot{\psi}_j} \right) d^4 x \right] \left| \det \left(\frac{\partial^2 L}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| D\phi D\dot{\psi}. \end{aligned} \quad (44)$$

VII. STRINGS

The tau or time derivatives of the coordinate fields X^μ in the Nambu-Gotō Lagrange density

$$L = -\frac{T_0}{c} \int_0^{\sigma_1} \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \quad (45)$$

do not occur quadratically. The momenta are

$$\mathcal{P}_\mu^\tau = \frac{\partial L}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \quad (46)$$

and the second derivatives of the Lagrange density are [8]

$$\begin{aligned} \frac{\partial^2 L}{\partial \dot{X}^\mu \partial \dot{X}^\nu} &= \frac{T_0}{c} \left[\frac{\delta_{\mu\nu} X'^2 - X'_\mu X'_\nu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \right. \\ &\quad \left. - \frac{\left((\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu \right) \left((\dot{X} \cdot X') X'_\nu - (X')^2 \dot{X}_\nu \right)}{\left[(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2 \right]^{3/2}} \right]. \end{aligned} \quad (47)$$

The proposed path integral (12) for the Nambu-Goto action is then

$$\begin{aligned} \langle X'' | e^{-i\tau H} | X' \rangle = & \int \exp \left[\int i(\dot{X}^\mu - \dot{Y}^\mu) \frac{\partial L(X, \dot{Y})}{\partial \dot{Y}^\mu} + iL(X, \dot{Y}) d^4x \right] \\ & \times \left| \det \left[\frac{\partial^2 L(X, \dot{Y})}{\partial \dot{Y}^\mu \partial \dot{Y}^\nu} \right] \right|_{DXD\dot{Y}} \end{aligned} \quad (48)$$

in which the formulas (46) and (47) (with $\dot{X}^\mu \rightarrow \dot{Y}^\mu$) are to be substituted for the first and second derivatives of the action density L with respect to the tau derivatives \dot{Y}^μ .

ACKNOWLEDGMENTS

Conversations with David Amdahl, Michael Grady, Sudhakar Prasad, and Shashank Shalgar advanced this work which arose out of numerical computations done with David Amdahl at the Korea Institute for Advanced Study and at the National Energy Research Scientific Computing Center, a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

-
- [1] P. Ade and others (Planck Collaboration), *Astron. Astrophys.* **566**, A54 (2014) (2013).
 - [2] S. Weinberg, *The Quantum Theory of Fields*, Vol. I (Cambridge University Press, 1995) Chap. 9; K. Cahill, *Physical Mathematics* (Cambridge University Press, 2013) Chap. 16.
 - [3] S. Boettcher and C. Bender, *J. Math. Phys.* **31**(11), 2579 (1990).
 - [4] K. Cahill, *Phys. Rev. D* **87**, 065024 (2013); *Phys. Rev. D* **88**, 125014 (2013).
 - [5] C. M. Bender and P. D. Mannheim, *Phys. Rev. Letters* **100**, 110402 (2008); *J. Phys. A* **41**, 304018 (2008).
 - [6] P. A. M. Dirac, *Can. J. Math.* **2**, 129 (1950); *Proc. Roy. Soc. London, ser. A* **246**, 326 (1958); *Lectures on Quantum Mechanics* (Yeshiva University and Dover Publications, 1964).
 - [7] K. Cahill, *Physical Mathematics* (Cambridge University Press, 2013) Chap. 14.
 - [8] K. Cahill, *Physical Mathematics* (Cambridge University Press, 2013) Chap. 19.