J. Phys. A: Math. Gen. 26 (1993) 7213–7217. Printed in the UK

Geometrical representations of gauge fields†

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Received 27 July 1993

Abstract. It has been known for some time that $U(n)$ gauge fields can be represented in terms of gradients of complex vectors with sufficiently many components. By using Monte Carlo simulations, one may determine how many components are needed for a given gauge group. Here we provide numerical evidence that a single three-component complex unit vector gives rise to the electromagnetic field and that two orthonormal five-component complex vectors result in $U(2)$.

1. Introduction

In the early 1960s Narasimhan and Ramanan [1] showed that every $U(n)$ gauge field $A^c_\mu(x)$ can be expressed in terms of n orthonormal vectors $e_a(x)$ in the form

$$\mathrm{i}t^c_{ab}A^c_\mu(x) = e^\dagger_a(x)\cdot e_{b,\mu}(x) \tag{1}$$

where the matrices t^c are the generators of $U(n)$ and the comma indicates partial differentiation with respect to x^μ. The n vectors $e_a(x)$ must have $N > n$ complex components in general. These vectors may be considered to be the columns of an $N \times n$ matrix $V(x)$

$$e_{ia}(x) = V_{ia}(x). \tag{2}$$

In terms of this matrix, the gauge field

$$A_\mu(x) = t^c A^c_\mu(x) \tag{3}$$

can be written as

$$A_\mu(x) = -\mathrm{i}V^\dagger(x)V_{,\mu}(x). \tag{4}$$

This relationship between vectors and gauge fields has been exploited by Corrigan *et al* [2] and by Atiyah [3] in their studies of classical solutions of the Yang–Mills equations. It has also been used by Dubois-Violette and Georgelin [4] as the basis of their formulation of gauge theory.

It is not, in general, known how many components the vectors e_a must have in order to represent the gauge fields of a given gauge group. In what follows we shall describe a numerical method for determining this minimum number N and shall report the results we obtained by applying this method to the groups $U(1)$ and $U(2)$. We have found that any

† Research supported by the US Department of Energy under contract DE-FG04-84ER40166.

0305-4470/93/237213+05$07.50

$U(1)$ gauge field can be represented by the motion of a single three-component complex unit vector $e(x)$ and that arbitrary $U(2)$ gauge fields can arise from the motion of two orthonormal five-component complex vectors $e_a(x)$.

We begin our discussion with two examples of the relationship (1) between vectors and gauge fields.

2. Illustrations

Our first example is that of a plane wave in electrodynamics. A single moving unit complex three-vector $e(x)$ gives rise to the gauge field

$$A_\mu(x) = -ie^\dagger(x) \cdot e_{,\mu}(x). \tag{5}$$

It is easy to verify that the vector

$$e(x) = \begin{pmatrix} e^{ip\cdot x} \cos k \cdot x \\ e^{iq\cdot x} \sin k \cdot x \\ 0 \end{pmatrix} \tag{6}$$

with $k^2 = 0$, $q^0 = p^0$, and $k \cdot (p - q) = 0$ represents a plane wave propagating in the k direction. The electric and magnetic fields of the wave are

$$E(x) = k^0(p - q) \sin 2k \cdot x \tag{7}$$

and

$$B(x) = k \times (p - q) \sin 2k \cdot x \tag{8}$$

respectively.

Our second example illustrates how the basic formula (1) can arise in a theory of free fermions. It is natural to write a multicomponent Fermi field

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \tag{9}$$

in terms of orthonormal basis vectors $e_a(x)$ in the coordinate-free form

$$\psi(x) = \psi_a(x) e_a(x). \tag{10}$$

The derivative in the free Dirac action density

$$\begin{aligned} \mathcal{L}_\mathcal{D} &= i\bar{\psi}\gamma^\mu \partial_\mu \psi = i\bar{\psi}_a e_a^\dagger \gamma^\mu \partial_\mu (\psi_b e_b) \\ &= i\bar{\psi}_a \gamma^\mu (\delta_{ab}\partial_\mu + e_a^\dagger \cdot (\partial_\mu e_b))\psi_b \end{aligned} \tag{11}$$

then becomes a covariant derivative

$$\begin{aligned} \mathcal{L}_\mathcal{D} &= i\bar{\psi}_a \gamma^\mu (\delta_{ab}\partial_\mu + ig t^c_{ab} A^c_\mu)\psi_b \\ &= i\bar{\psi}_a \gamma^\mu D^{ab}_\mu \psi_b \end{aligned} \tag{12}$$

with its gauge field A^c_μ given by the relation (1).

A complete gauge theory would include a Lagrange density for the gauge fields

$$\mathcal{L}_{\mathcal{YM}} = -\frac{1}{4g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} \tag{13}$$

in which $F^{\mu\nu}$ is the field-strength matrix

$$F_{\mu\nu} = \mathrm{i}[D_\mu, D_\nu] \\ = A_{\mu,\nu} - A_{\nu,\mu} - \mathrm{i}[A_\mu, A_\nu] \tag{14}$$

where A_μ is the gauge-field matrix (3). The orthonormality of the vectors e_a would have to be established either by an economical representation of these vectors or by Lagrange multipliers. Finally the measure for the path integral should embody the Jacobian of the transformation (1) from the vectors to the gauge fields.

3. How big is N?

Narasimhan and Ramanan [1] have shown that for the gauge group $U(n)$ in manifolds of d dimensions, the required number of rows N for the matrix $V(x)$ has the upper bound

$$N \leqslant (d+1)(2d+1)n^3. \tag{15}$$

They also showed that locally the bound on N is smaller,

$$N_{\mathrm{loc}} \leqslant (2d+1)n^3. \tag{16}$$

Dubois-Violette and Georgelin [4] have given a lower bound for the minimum number N of rows for the group $U(n)$. By counting the independent functions required by the gauge field and available in the components of the matrix $V(x)$, they noted that N_{loc} has the lower bound

$$N_{\mathrm{loc}} \geqslant (d+1)n/2. \tag{17}$$

For $U(1)$ Dubois-Violette and Georgelin showed that number N of rows of the matrix V locally need only be

$$N_{\mathrm{loc}} = [(d+1)/2] \tag{18}$$

i.e. the least integer not less than $(d+1)/2$. In four dimensions, this value of N_{loc} is 3. Our numerical work confirms their result.

For $U(2)$ in four dimensions, their lower bound (17) is

$$N_{\mathrm{loc}} \geqslant 5. \tag{19}$$

Our work indicates that $N_{\mathrm{loc}} = 5$.

4. A Monte Carlo technique

For the purposes of field theory in flat space, it is probably sufficient to know N_{loc}. This integer can be determined for a given unitary group $U(n)$ by a method inspired by lattice gauge theory. One replaces the spacetime continuum by a four-dimensional lattice and puts the gauge fields on the links of the lattice and the vectors with N_{test} complex components on the vertices of the lattice.

In each Monte Carlo run, one starts with a random target configuration of gauge fields and random initial orthonormal vectors. One allows the vectors to evolve randomly and computes at each step the gauge fields generated by the vectors according to a discrete version of the relation (1), accepting those changes in the vectors that bring the generated gauge fields closer to the target gauge fields. If after many thousands of sweeps through the lattice, the difference between the generated and target gauge fields approaches zero and if this behaviour is exhibited in several runs, then one may tentatively conclude that $N_{\mathrm{loc}} \leqslant N_{\mathrm{test}}$.

5. Our numerical work

We have carried out numerical experiments that indicate that the motion of a single three-component complex unit vector $e(x)$ can be interpreted as a $U(1)$ gauge field

$$A_\mu(x) = -ie^\dagger(x) \cdot e_{,\mu}(x) \tag{20}$$

and that the motion of two five-component complex orthonormal vectors $e_a(x)$ yields the $U(2)$ gauge fields

$$A^j_\mu(x) = -i\sigma^j_{ba} e^\dagger_a(x) \cdot e_{b,\mu}(x) \tag{21}$$

in which the 2×2 matrices $\sigma^j = (1, \sigma)$ are the identity matrix and the Pauli matrices σ.

In our experiments we used a periodic spacetime lattice of size 10^4 with vectors $e_a(m)$ on the vertices m of the lattice and gauge fields on the links l of the lattice.

We began each run with random initial orthonormal vectors $e^0_a(m)$ and with random target gauge fields $\mathcal{A}^c_\mu(l)$ whose values we uniformly distributed over the interval $[-0.3, 0.3]$.

The vectors $e_a(m)$ generated gauge fields $A^c_\mu(l)$ according to the equation

$$A^c_\mu(l) = \frac{-i}{k} t^c_{ba} e^\dagger_a(m) \cdot (e_b(m + \hat{\mu}) - e_b(m)) \tag{22}$$

which is a discrete version of the basic relation (1). Here the link l joins the vertex m to the vertex $m + \hat{\mu}$ which lies one lattice spacing beyond m in the μth direction. We allowed the vectors to evolve randomly and selected those mutations that lowered the sum D of the squares of the differences

$$D = \sum_{l,c} (\mathcal{A}^c_\mu(l) - A^c_\mu(l))^2. \tag{23}$$

For the case of $U(1)$, we made two long runs in which the unit vectors had three complex components. In the first run, the initial value of D was 1208. After 20 000 sweeps through the lattice, the final value of the difference D was 0.001 37. In the second run, the initial value of D was 1196. After 20 000 sweeps through the lattice, the final value of the difference D was 0.001 40. We conclude that an arbitrary $U(1)$ gauge field can be described by the motion of a single three-component complex unit vector $e(x)$.

For the case of $U(2)$, we made two long runs in which the orthonormal vectors had five complex components. In the first run, the initial value of D was 4788. After 200 000 sweeps through the lattice, the final value of the difference D was 0.2685. In the second run, the initial value of D was 4790. After 200 000 sweeps through the lattice, the value of the difference D was 0.2709; after 300 000 sweeps, it was 0.1605; and after 400 000 sweeps, it was 0.1122. We conclude that an arbitrary $U(2)$ gauge field can be described by the motion of two orthonormal five-component complex vectors $e_a(x)$.

It is easy to explain why the $U(1)$ vector converges much faster than the $U(2)$ vectors. For $U(1)$ the gauge field has four independent functions, while the unit complex three-vector $e(x)$ has five. But for $U(2)$ both the gauge fields and the two orthonormal complex five-vectors have 16 independent functions. Thus the fit is loose for $U(1)$ and tight for $U(2)$.

Acknowledgements

We are grateful to S L Adler, G H Herling, B Kemmell, R Reeder and K Uhlenbeck for helpful conversations and to the US Department of Energy for their support of this

work under grant DE-FG04-84ER40166. The computations reported here were performed on Cray's at the National Energy Research Supercomputer Center of the Department of Energy, on a DECstation 3100, and on a DEC Alpha 3000/400.

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