Z. Phys. C - Particles and Fields 29, 537-539 (1985)

Zeitschrift für Physik C Particles and Fields

The Strong CP Problem in a Compact Robertson-Walker Universe[1]

Kevin Cahill

Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131, USA

Received 25 July 1985

Abstract. For a universe of the compact Robertson-Walker cosmology, Gauss's law requires the QCD vacuum angle θ to vanish, taking with it the strong CP problem.

The strong interactions conserve C, P, and T as far as we know. But due to the structure of the vacuum, the P- and T-violating term $E_i^a B_i^a$ occurs in the lagrangian of quantum chromodynamics multiplied by the arbitrary vacuum angle θ. This conflict between experiment and theory is known as the strong CP problem [1].

The purpose of the present note is to show that for a universe of the compact Robertson-Walker cosmology, Gauss's law requires the vacuum angle θ to vanish. In this Robertson-Walker cosmology, in which $k=+1$, the spatial universe at every moment of time is a three-dimensional spherical surface, S^3. For this cosmology, and presumably for a class of similar cosmologies, Gauss's law solves the strong CP problem. However Gauss's law does not prevent the quark mass matrix from producing unwanted P and T violations.

It may seem bizarre to suggest that the curvature of spacetime on a scale much larger than that of the strong interactions could have anything to do with the strong CP problem. However the problematic term $\theta E_i^a B_i^a$ is a total divergence; so the nature of the boundary of spacetime is relevant.

Let us recall the origin of the term $\theta E_i^a B_i^a$ in flat space. In Dirac's method of quantization, the constraint known as Gauss's law is represented by the requirement that the operator $D_i^{ac} F_c^{0i} - j^{0a}$ annihilate all physical states [2]. By multiplying this operator by a gauge parameter, $\omega_a(x)$, and integrating over space, one may conclude that the operator

$$\mathrm{Gauss}(\omega) = \int d^3x\, \omega_a [D_i^{ac} F_c^{0i} - j^{0a}] \tag{1}$$

also annihilates all physical states. This operator differs from the generator of the gauge transformation associated with ω

$$\mathrm{Gauge}(\omega) = \int d^3x [-F_a^{0i} D_i^{ac} \omega_c - \omega_a j^{0a}] \tag{2}$$

by the surface term $\Sigma = \int d\sigma_i \omega_a F_a^{0i}$. This surface term is zero for gauge parameters that vanish at spatial infinity. Thus physical states are invariant under "little" gauge transformations. However, physical states are not required by Gauss's law to be invariant under a gauge transformation for which the gauge parameter $\omega_a(x)$ does not vanish at spatial infinity. Such "big" gauge transformations can be grouped into homotopy classes labeled by the integers. Since even big gauge transformations leave the hamiltonian invariant, the vacuum can be chosen to be a simultaneous eigenstate of both the hamiltonian and of a big gauge transformation that takes a pure gauge field from the nth homotopy class to the $n+1$st. Because gauge transformations are implemented by unitary transformations, the eigenvalue of the vacuum is unimodular, $\exp(i\theta)$. The phase θ is the QCD vacuum angle. The action functional for the θ vacuum differs from the one for the $\theta=0$ vacuum by an integral over spacetime of a term proportional to $\theta E_i^a B_i^a$.

It is intuitively clear that the surface term Σ may play a different role in a world without a spatial boundary. Let us represent the gravitational field by a classical metric field, $g_{\mu\nu}(x)$, and let $h(x)$ be the square-root of the absolute value of the determinant of this metric. Then the operator Gauss(ω) appro-

[1] Supported in part by the U.S. Department of Energy under grant DE-FG04-84ER40166

priate to this background metric is

$$\mathrm{Gauss}(\omega)=\int d^3x h\omega_a[h^{-1}D_i^{ac}hF_c^{0i}-j^{0a}], \tag{3}$$

in which the sigma or gamma matrix in the quark current is suitably modified by the vierbein of the metric. The physical states are annihilated by this operator for all ω. The generator of the gauge transformation associated with the gauge parameter ω is

$$\mathrm{Gauge}(\omega)=\int d^3x h[-F_a^{0i}D_i^{ac}\omega_c-\omega_a j^{0a}]. \tag{4}$$

As in flat space, the two differ by the surface term $\Sigma=\int d\sigma_i h\omega_a F_a^{0i}$. But now if the space described by the metric $g_{\mu\nu}$ has no spatial boundary, then Σ may vanish even if ω_a is multivalued. In such worlds Gauss's law requires the physical states to be invariant under all gauge transformations. The vacuum angle θ therefore vanishes, and there is no strong *CP* problem. It will now be shown that this is the case for the compact Robertson-Walker cosmology.

The sphere S^3 may be parameterized by the angles θ, ϕ, and χ which run from 0 to π, 0 to π, and 0 to 2π. In terms of these variables, the metric of the sphere is diagonal with elements $g_{\theta\theta}=1$, $g_{\phi\phi}=\sin^2\theta$, and $g_{\chi\chi}=\sin^2\theta\sin^2\phi$. The scale factor h is then proportional to $\sin^2\theta\sin\phi$. Let us first consider the gauge group SU_2 an arbitrary element of which may be represented in terms of the Pauli matrices as $U=\exp(i\omega_a\sigma_a/2)$. The map defined by $\omega_1=2\theta\sin\phi\sin(n\chi)$, $\omega_2=2\theta\sin\phi\cos(n\chi)$, and $\omega_3=2\theta\cos\phi$ from S^3 to SU_2 lies in the nth homotopy class*. Because S^3 has no boundary, the surface term Σ would vanish trivially if the gauge parameter ω were single-valued. However since the pairs of points (π,ϕ,χ) and (π,ϕ',χ') are to be identified, ω is multivalued. The surface term Σ consists of three terms. The first term is an integral over θ and ϕ of the difference between $hE_\chi^a\omega_a$ evaluated at $\chi=2\pi$ and at $\chi=0$. This term vanishes because ω is single-valued as a function of χ and because E, like U, is single-valued. The second term is an integral over θ and χ of the difference between $hE_\phi^a\omega_a$ evaluated at $\phi=\pi$ and at $\phi=0$. This term vanishes because $h=0$ for $\phi=0$ and for $\phi=\pi$. The third term is an integral over ϕ and χ of the difference between $hE_\theta^a\omega_a$ evaluated at $\theta=\pi$ and at $\theta=0$. This term vanishes because $h=0$ for $\theta=\pi$ and for $\theta=0$. Thus $\Sigma=0$ for all n.

Let us now consider the case of an arbitrary connected compact Lie group G of which the general element U can be represented in terms of the generators t_a of G as $U=\exp(i\omega_a t_a)$. The gauge parameter ω will be some multivalued function of θ, ϕ, and χ such that the map from S^3 to G is single-valued and lies in some homotopy class. The surface term Σ will again consist of three terms similar to those for $G=SU_2$. Once again the second and third terms will vanish because of the metric factor h. The first term will vanish if ω is single-valued as a function of χ, as in the case of SU_2. Since the points $(\theta,\phi,0)$ and $(\theta,\phi,2\pi)$ are to be identified, the group element U must be continuous across the corresponding $\theta-\phi$ plane. This means that after a suitable unitary transformation, the matrices $\omega(\theta,\phi,0)_a t_a$ and $\omega(\theta,\phi,2\pi)_a t_a$ must differ by a diagonal matrix whose eigenvalues are integral multiples of $2\pi i$. By continuity these eigenvalues cannot change with θ and ϕ, although the unitary transformation can. Thus close to the pole $\theta=0$ these eigenvalues must still be the same multiples of $2\pi i$. But here the two points $(\theta,\phi,0)$ and $(\theta,\phi,2\pi)$ can be connected by a very short loop around the pole, a loop that does not cross the $\theta-\phi$ plane on which ω is multivalued. On this loop the matrix $\omega_a t_a$ must race from its value at $\chi=0$ to its value at $\chi=2\pi$ forcing its exponential to gyrate wildly. The map from S^3 to G is therefore discontinuous unless ω is singlevalued on the $\theta-\phi$ plane, in which case the first term of Σ also vanishes.

These arguments can no doubt be improved and extended to a broader class of *CP*-free cosmologies. They do not apply, however, to the physically irrelevant case of the circle because it is doubly connected and flat.

Another source of *CP* violation is the quark mass matrix, which may contain pieces that are imaginary, off-diagonal, or that contain γ_5. Gauss's law does not cure these sources of *CP* violation, which are also required to be small. However the Higgs sector of the theory is itself so arbitrary that the added requirement that the quark mass matrix be real, diagonal, and free of γ_5 seems to increase its arbitrariness only marginally.

The classic solution to the strong *CP* problem, and also to the Higgs *CP* problem, involves a $U(1)$ symmetry broken by instantons and is due to Peccei and Quinn [3]. However their solution entails the existence of axions [4], which are hypothetical light, weakly interacting, pseudoscalar bosons. Axions have been searched for without success in K, Ψ, and Υ decays and in reactor and beam-dump experiments. An astrophysical argument (lest axions freeze stars [5]) and a cosmological one (lest axions overclose the universe [6]), together with the null results of these axion searches, seem to limit possible axion masses to the range $1.2\times10^{-5}\,\mathrm{eV}\leqq m_a\leqq0.13\,\mathrm{eV}$ [7]. A search of this range has been proposed [8] and would be of considerable interest.

A second possible explanation of the strong *CP* problem, and the Higgs one, is the existence of a

* At $\theta=\pi$, $U=-1$ and $h=0$.

massless quark [4]. However this explanation seems to disagree with current algebra [4].

We are therefore faced with the following alternatives: either (1) there are axions or (2) there is a massless quark or (3) the spatial universe is S^3 or some other *CP*-free space or (4) there is another resolution to the strong *CP* problem. A number of recent candidates for option (4) are listed in [9].

Acknowledgments. The author is grateful to M. Buchner, J.D. Finley, R. Hersh, and R. Reeder for useful conversations. Some of this work was done at the Aspen Center for Physics and at SLAC. This work was supported by the Department of Energy under grant DE-FG04-84ER40166.

References

1. G. 'tHooft: Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. **D14**, 3432 (1976); R. Jackiw, C. Rebbi: Phys. Rev. Lett. **37**, 172 (1976); C.G. Callan, R.F. Dashen, D.J. Gross: Phys. Lett. **63B**, 334 (1976)
2. P.A.M. Dirac: Canad. J. Math. **3**, 1 (1951); Lectures on Quantum Mechanics, pp. 1–43. Belfer Graduate School of Science, Yeshiva U., N.Y. (1964); A. Hanson, T. Regge, C. Teitelboim: Constrained Hamiltonian Systems, pp. 71–101. Accademia Nazionale dei Lincei, Rome (1976)
3. R.D. Peccei, H.R. Quinn: Phys. Rev. Lett. **38**, 1440 (1977); Phys. Rev. **D16**, 1791 (1977)
4. S. Weinberg: Phys. Rev. Lett. **40**, 223 (1978); F. Wilczek: Phys. Rev. Lett. **40**, 279 (1978)
5. D. Dicus, E. Kolb, V. Teplitz, R. Wagoner: Phys. Rev. **D18**, 1829 (1978); **D22**, 839 (1980); M. Fukugita, S. Watamura, M. Yoshimura: Phys. Rev. Lett. **48**, 1522 (1982)
6. L. Abbott, P. Sikivie: Phys. Lett. **120B**, 133 (1983); J. Preskill, M. Wise, F. Wilczek: Phys. Lett. **120B**, 127 (1983); M. Dine, W. Fischler: Phys. Lett. **120B**, 137 (1983)
7. P. Sikivie: U. of Florida preprint UFTP-84-5
8. P. Sikivie: Phys. Rev. Lett. **51**, 1415 (1983); **52**, 695 (1983); J.E. Moody, F. Wilczek: Santa Barbara preprint ITP-83-177; P. Sikivie, N. Sullivan, D. Tanner: unpublished
9. S.M. Barr: Phys. Rev. **D30**, 1805 (1984); A. Nelson: Phys. Lett. **134B**, 422 (1984); **136B**, 387 (1984); Harvard preprint HUTP-84/A022; M. Dugan, B. Grinstein, L. Hall: Harvard preprint HUTP-84/A023; H. Levine, S.B. Libby: Brown preprint HET-539; H. Georgi, A. Nelson, M. Shin: Harvard preprint HUTP-84/A071