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Tensor gauge fields and dark matter in general relativity with fermions

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Abstract

The action of general relativity with fermions has two independent symmetries: general coordinate invariance and local Lorentz invariance. General coordinate transformations act on coordinates and tensor indices, while local Lorentz transformations act on Dirac and Lorentz indices, much like a noncompact internal symmetry.

The internal-symmetry character of local Lorentz invariance suggests that it might be implemented by tensor gauge fields with their own Yang–Mills action rather than by the spin connection as in standard formulations. But because the Lorentz group is noncompact, their Yang–Mills action must be modified by a neutral vector field whose average value at low temperatures is timelike. This vector field and the tensor gauge fields are neutral and interact gravitationally, so they contribute to hot and cold dark matter.

The two independent symmetries of the action are reduced to a single symmetry of the vacuum, local Lorentz invariance, by the nonzero average values of the tetrads $c^a_{\ k}$. The local Lorentz invariance of general relativity with fermions can be extended to local U(2, 2) invariance.

If the contracted squares of the covariant derivatives of the tetrads multiplied by the square of a mass M are added to the action, then in the limit $M^2 \to \infty$, the equation of motion of the tensor gauge fields is the vanishing of the covariant derivatives of the tetrads, which is Cartan's first equation of structure. In the same limit, the tensor gauge fields approach the spin connection.

Keywords: general relativity, cosmology, fermions, dark matter, tensor gauge fields, gauge theory

1. Introduction

The action of general relativity with fermions has two independent symmetries: general coordinate invariance and local Lorentz invariance. These symmetries are traditionally implemented by Cartan's tetrads c^a_i and by the spin connection ω_i which is a quartic polynomial in the tetrads and their first derivatives. In this paper, the spin connection is replaced by tensor gauge fields with their own Yang–Mills action.

General coordinate invariance is the defining symmetry of Einstein's general relativity. A general coordinate transformation $x \to x'$ acts on coordinates x^i and on tensor indices i, k but leaves Dirac indices α, β and Lorentz indices a, b unchanged

$$\psi'_\alpha(x') = \psi_\alpha(x), \quad c'_a{}^i(x') = \frac{\partial x'^i}{\partial x^k}\, c_a{}^k(x). \tag{1}$$

General coordinate invariance is implemented by Cartan's tetrads c^a_k and their derivatives.

Local Lorentz transformations act on Dirac and Lorentz indices but leave coordinates and tensors unchanged

$$\begin{aligned} \psi'_\alpha(x) &= D_{\alpha\beta}(\Lambda(x))\, \psi_\beta(x) \\ (D_i\psi)'_\alpha &= (\partial_i + \omega'_i)^{\alpha\beta}\psi'_\beta = D_{\alpha\beta}(\Lambda)\,(D_i\psi)_\beta \\ c'^a{}_i(x) &= \Lambda^a{}_b(x)\, c^b{}_i(x). \end{aligned} \tag{2}$$

Local Lorentz invariance is implemented by the spin connection ω_i in standard formulations [1–7].

Invariance under general coordinate transformations and invariance under local Lorentz transformations are both exact and independent symmetries of the action of general relativity with fermions. But while general coordinate transformations (1) act on coordinates and tensor indices, local Lorentz transformations (2) act on Lorentz and Dirac indices leaving coordinates unchanged. In this respect, local Lorentz invariance is like a noncompact internal symmetry [8].

These observations motivated an attempt [9] to implement local Lorentz invariance by means of tensor gauge fields $L^{ab}{}_i$, but the focus was so exclusively upon fermions and the spin connection, that the invariance of the action of the tensor gauge fields was neglected and repaired belatedly in an erratum [10]. In that erratum, a hermitian matrix $h(x)$ was introduced that under a local Lorentz transformation $\Lambda(x)$ transforms as $h'(x) = D^\dagger(\Lambda(x))h(x)D(\Lambda(x))$ in which $D(\Lambda)$ is Dirac's $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $SO(3,1)$.

In the present paper, local Lorentz invariance is implemented by means of tensor gauge fields L^{ab}_i and a real vector field K_i in terms of which the matrix h is realized as $h = \mathrm{i}\beta\gamma^a c_a{}^i K_i = \mathrm{i}\beta\gamma^a K_a$. The spin connection $\omega_i = \frac{1}{8}\omega^{ab}{}_i[\gamma_a, \gamma_b]$ is replaced by a 'Lorentz connection' $L_i = \frac{1}{8}L^{ab}{}_i[\gamma_a, \gamma_b]$ with field-strength $F_{ik} = [\partial_i + L_i, \partial_k + L_k]$ and Yang–Mills-like action

$$S_L = -\frac{1}{4m^2\lambda^2}\int \mathrm{Tr}(F^\dagger_{ik}\, hF^{ik}\, \beta h\beta)\sqrt{g}\; \mathrm{d}^4x \tag{3}$$

in which $\beta = i\gamma^0$, $g = |\det(g_{ik})|$, and λ is a coupling constant. The vector field $K_i(x)$ makes the trace in the action S_L invariant under noncompact Lorentz transformations, but the squares of the time derivatives $(\dot{L}^{ab}{}_i)^2$ appear in S_L with positive signs only if the average value of K_i is timelike.

The action of the vector boson K_i is

$$S_K = \int \left[-\frac{1}{4}(D_i K_k - D_k K_i)(D^i K^k - D^k K^i) - \frac{1}{4}\xi^2 (K_i K^i + m^2)^2 \right] \sqrt{g}\ d^4x \tag{4}$$

in which $\xi > 0$ is a positive coupling constant. The potential energy density $\frac{1}{4}\xi^2(K_i K^i + m^2)^2$ makes the average value $\langle 0|K_i(x)|0\rangle$ of $K_i(x)$ timelike at low temperatures.

At low temperatures, $T \ll \xi m$, the vibrations about $\langle 0|K_i(x)|0\rangle$ are massless and neutral, and so would contribute to hot dark matter. But at high temperatures, $T \gg \xi m$, the field K_i radiates particles of mass $m_K = \xi m$ which would contribute to cold dark matter at low temperatures.

The tensor gauge fields $L^{ab}{}_i$ are neutral and massless and would contribute to hot dark matter.

Since the tensor gauge fields $L^{ab}{}_i$ and the vector boson K_i interact with gravitational strength, they would have decoupled much earlier than the photons, and so would not have been heated by the annihilations of the quarks and leptons. They would have a present temperature much colder than the 2.7 K of the CMB, which decoupled when the present universe was 380 000 years old and had a temperature of about 0.26 eV.

In terms of the connection L_i and the gauge fields of the standard model $A_i = \mathrm{i}A^{\alpha}_{is}\, t^{\alpha s}$, the covariant derivative of a Dirac field ψ is defined in this paper as

$$D_i\psi = (\partial_i + L_i + A_i)\psi \tag{5}$$

rather than in terms of the spin connection as

$$D_i\psi = (\partial_i + \omega_i + A_i)\psi \tag{6}$$

in which $\omega_i = \frac{1}{8}\omega^{ab}{}_i[\gamma_a, \gamma_b]$ and [1–7]

$$\omega^{ab}{}_i = \tfrac{1}{2}c^{aj}(\partial_i c^b{}_j - \partial_j c^b{}_i) - \tfrac{1}{2}c^{bj}(\partial_i c^a{}_j - \partial_j c^a{}_i) - \tfrac{1}{2}c^{ak}c^{b\ell}c^c{}_i(\partial_k c_{c\ell} - \partial_\ell c_{ck}). \tag{7}$$

The present formalism has a serious disadvantage: it introduces one vector boson K_i and six tensor gauge fields $L^{ab}{}_i$. On the other hand, it has three advantages:

1. It treats local Lorentz invariance as an internal symmetry and gives it a Yang–Mills action.
2. It adds to the usual internal-symmetry gauge fields $A_i = \mathrm{i}A^{\alpha}_{is}\, t^{\alpha s}$ in the Dirac covariant derivative $D_i\psi = (\partial_i + L_i + A_i)\psi$ a linear combination of gauge fields $L_i = \frac{1}{8}L^{ab}{}_i[\gamma_a, \gamma_b]$ and not a quartic polynomial ω_i in the tetrads and their derivatives.
3. At high temperatures, the vector field K radiates massive particles that would contribute to cold dark matter at low temperatures.

The action S_L is discussed in section 2. The matrix h, the vector K_i, and the positive signs of squares of the time derivatives $(\dot{L}^{ab}{}_i)^2$ are discussed in section 3. The Dirac action

$$S_D = -\int \bar{\psi}\ \gamma^a c_a{}^i D_i \psi \sqrt{g}\ \mathrm{d}^4x \tag{8}$$

in which D_i is the covariant derivative (5) is discussed in section 4. The actions S_L, S_K and S_D are invariant under local Lorentz transformations and under independent general coordinate transformations.

Although general coordinate invariance and local Lorentz invariance are independent symmetries of the action, they are not independent symmetries of the ground state of the Universe because they do not leave invariant the nonzero average values of Cartan's tetrads $c^a{}_k$. Their average values $\langle 0|c^a{}_k|0\rangle$ or $\mathrm{Tr}(\rho\ c^a{}_i(x))$ reduce the symmetries of the action—general coordinate invariance and local Lorentz invariance—to a single symmetry of the ground state: local Lorentz invariance. Since nonzero average tetrad values are intrinsic to the theory, this reduction of symmetry is intrinsic rather than spontaneous. It is discussed in section 5.

Local invariance under the Lorentz group SO(3, 1) is extended to U(2, 2) in section 6.

If tensor gauge fields do gauge SO(3,1) and if all invariant terms of dimension (mass)^4 or less occur in the action (which is not obvious in a theory of gravity), then the total action would include the contracted squares

$$S_C = -M^2 \int D_i c^a{}_k D^i c_a{}^k \sqrt{g}\ \mathrm{d}^4 x \tag{9}$$

of the covariant derivatives of the tetrads

$$D_i c^a{}_k = \partial_i c^a{}_k + L_i{}^a{}_b c^b{}_k - \Gamma^\ell{}_{ki}\, c^a{}_\ell. \tag{10}$$

The factor M^2 is required because tetrads are dimensionless. The linear combination $L_i{}^a{}_b c^b{}_k - \Gamma^\ell{}_{ki}\, c^a{}_\ell$ of the tensor gauge field and the Levi-Civita connection acquires a mass of order M. In the limit $M \to \infty$ the equation of motion of the tensor gauge fields $L^{ab}{}_i$ arises from the term S_C and is $D_i c^a{}_k = 0$ which is Cartan's first equation of structure. In the same limit, the tensor gauge fields approach the spin connection as described in section 7.

2. Action of tensor gauge bosons

The action (3) proposed for the tensor gauge fields is

$$S_L = -\frac{1}{4m^2\lambda^2} \int \mathrm{Tr}\,(F^\dagger_{ik}\ h\ F^{ik}\ \beta h \beta)\sqrt{g}\ \mathrm{d}^4 x \tag{11}$$

in which the field strength F_{ik} is

$$F_{ik} = [\partial_i + L_i,\ \partial_k + L_k], \tag{12}$$

the matrix of tensor gauge fields L_i is

$$L_i = \tfrac{1}{8}\, L^{ab}{}_i [\gamma_a,\ \gamma_b], \tag{13}$$

$h = i\beta\gamma^a c_a{}^i K_i$ is a 4×4 hermitian matrix, K_i is a real vector field, and λ is a coupling constant.

The action (11) is real because

$$\mathrm{Tr}(F^\dagger_{ik} h F^{ik} \beta h \beta)^* = \mathrm{Tr}(\beta h \beta\ F^{\dagger ik}\ h F_{ik}) = \mathrm{Tr}(F^\dagger_{ik}\ h F^{ik}\ \beta h \beta). \tag{14}$$

The gamma matrices

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = i\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\beta\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\gamma^i = -i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i\beta\boldsymbol{\gamma} = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

asatisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab} I$. The commutators $[\gamma_a, \gamma_b]$ in $L_i = -\frac{1}{8} L^{ab}{}_i [\gamma_a, \gamma_b]$ are for spatial a, b, $c = 1, 2, 3$

$$[\gamma_a,\ \gamma_b] = 2i\epsilon_{abc}\sigma^c I \quad \text{and} \quad [\gamma_0,\ \gamma_a] = -2\sigma^a\gamma^5. \tag{16}$$

The gauge fields associated with rotations and boosts are

$$\boldsymbol{r}^a_i \equiv \frac{1}{2}\epsilon_{abc} L^{bc}{}_i \quad \text{and} \quad \boldsymbol{b}^a_i \equiv L^{a0}{}_i, \tag{17}$$

and the matrix of gauge fields L_i is

$$L_i = -i\,\frac{1}{2}\,\boldsymbol{r}_i \cdot \boldsymbol{\sigma}\, I - \frac{1}{2}\,\boldsymbol{b}_i \cdot \boldsymbol{\sigma}\gamma^5. \tag{18}$$

The field strength (12) is then

$$F_{ik} = [\partial_i + L_i, \partial_k + L_k] = -i\frac{1}{2}[\partial_i \boldsymbol{r}_k - \partial_k \boldsymbol{r}_i + (\boldsymbol{r}_i \times \boldsymbol{r}_k - \boldsymbol{b}_i \times \boldsymbol{b}_k)] \cdot \boldsymbol{\sigma} I$$
$$- \tfrac{1}{2}[\partial_i \boldsymbol{b}_k - \partial_k \boldsymbol{b}_i + (\boldsymbol{r}_i \times \boldsymbol{b}_k + \boldsymbol{b}_i \times \boldsymbol{r}_k)] \cdot \boldsymbol{\sigma}\gamma^5. \tag{19}$$

Under a local Lorentz transformation $D = D(\Lambda(x))$, these fields transform as

$$\begin{aligned}
L'^{ab}{}_i &= \Lambda^a{}_c \Lambda^b{}_d L^{cd}{}_i - \tfrac{1}{2}\,\mathrm{Tr}\,(D\partial_i D^{-1}[\gamma^a, \gamma^b]) \\
\partial_i + L'_i &= D(\partial_i + L_i)\, D^{-1} \\
F'_{ik} &= D\, F_{ik}\, D^{-1} \\
h' &= D^{-1\dagger}\, h\, D^{-1} \\
(\beta h \beta)' &= D\, \beta h \beta\, D^\dagger \\
\psi' &= D\, \psi
\end{aligned} \tag{20}$$

and so the action density s_L of the action S_L (11) is invariant

$$s'_L = \mathrm{Tr}(F'^{\dagger}_{ik}\, h' F'^{ik}\, \beta h' \beta) = \mathrm{Tr}(D^{-1\dagger} F^\dagger_{ik} D^\dagger\, D^{-1\dagger} h D^{-1} D F_{ik} D^{-1}\, D\beta h \beta D^\dagger) = \mathrm{Tr}(F^\dagger_{ik}\, h F^{ik}\, \beta h \beta) = s_L \tag{21}$$

under local Lorentz transformations (2) as well as under general coordinate transformations (1).

The squares of the time derivatives $(\dot{L}^{ab}{}_i)^2$ and $(\dot{L}^{a0}{}_i)^2$ of the gauge fields must appear with a positive sign in the action S_L (11) if the gauge-field action is to be bounded below. They will appear with a positive sign at low temperatures if the vector boson K_i in the matrix $h = \mathrm{i}\beta\gamma^a c_a{}^i K_i$ has an average value $\langle 0|K_i|0\rangle = K_{0i}$ in the low-temperature ground state that is timelike, $K_{0i} K^i_0 \simeq -m^2 < 0$. The matrix h, the vector K_i, and the signs of the time derivatives $(\dot{L}^{ab}{}_i)^2$ and $(\dot{L}^{a0}{}_i)^2$ are discussed in section 3.

The tensor gauge fields L^{ab}_i have spin 2 (not 3) because they are antisymmetric in a and b.

An explicit formula for the matrix $D(\Lambda)$ is given in appendix A along with derivations of the identities

$$D^\dagger \beta D = \beta \quad \text{and} \quad D\beta D^\dagger = \beta. \tag{22}$$

These identities (22) imply that the action S_L with h replaced by β, i.e the trace $\mathrm{Tr}\,(F^\dagger_{ik}\beta F^{ik}\beta)$, is invariant under local Lorentz transformations. But the choice $h \to \beta$ gives an action in which the squares of the time derivatives of the tensor gauge fields that gauge boosts and those that gauge rotations occur with opposite signs. So the trace (11) which uses the matrix $h = i\beta\gamma^a K_a$ may be the only viable choice.

With the abbreviations

$$\begin{aligned}
\boldsymbol{R}_{ik} &= \partial_i \boldsymbol{r}_k - \partial_k \boldsymbol{r}_i + (\boldsymbol{r}_i \times \boldsymbol{r}_k - \boldsymbol{b}_i \times \boldsymbol{b}_k) \\
\boldsymbol{B}_{ik} &= \partial_i \boldsymbol{b}_k - \partial_k \boldsymbol{b}_i + (\boldsymbol{r}_i \times \boldsymbol{b}_k + \boldsymbol{b}_i \times \boldsymbol{r}_k),
\end{aligned} \tag{23}$$

the field strength F_{ik} is $F_{ik} = -i\frac{1}{2}\boldsymbol{R}_{ik} \cdot \boldsymbol{\sigma} I - \frac{1}{2}\boldsymbol{B}_{ik} \cdot \boldsymbol{\sigma}\,\gamma^5$, and the action S_L is

$$S_L = -\frac{1}{16 m^2 \lambda^2} \int \mathrm{Tr}[(\boldsymbol{R}_{ik} \cdot \boldsymbol{\sigma} I + \mathrm{i}\boldsymbol{B}_{ik} \cdot \boldsymbol{\sigma}\gamma^5) h (\boldsymbol{R}^{ik} \cdot \boldsymbol{\sigma} I - \mathrm{i}\boldsymbol{B}^{ik} \cdot \boldsymbol{\sigma}\gamma^5)\beta h \beta]\sqrt{g}\; \mathrm{d}^4 x. \tag{24}$$

Tensor gauge fields $L^{ab}_{\ \ i}$ possess at least two other actions that are invariant under local Lorentz transformations and general coordinate transformations. One is succinct and linear

$$S_E = M_E^2 \int F^{ab}_{\ \ ik}\, c_a^{\ i}\, c_b^{\ k} \sqrt{g}\, \mathrm{d}^4x \tag{25}$$

in the coefficients $F^{ab}_{\ \ ik}$ of the field strength

$$F_{ik} = F^{ab}_{\ \ ik}[\gamma_a, \gamma_b] = \partial_i L_k - \partial_k L_i + [L_i, L_k]. \tag{26}$$

These coefficients

$$F^{ab}_{\ \ ik} = \partial_i L^{ab}_{\ \ k} - \partial_k L^{ab}_{\ \ i} + L^{bc}_{\ \ i} L^{a}_{\ ck} - L^{ac}_{\ \ i} L^{b}_{\ c\,k}$$

resemble Riemann's curvature tensor

$$R^k_{\ i\ell n} = \partial_\ell \Gamma^k_{\ in} - \partial_n \Gamma^k_{\ i\ell} + \Gamma^k_{\ m\ell}\, \Gamma^m_{\ in} - \Gamma^k_{\ mn}\, \Gamma^m_{\ i\ell}. \tag{27}$$

But the action S_E (25) does not lead to second-order differential equations for the gauge fields $L^{ab}_{\ \ i}$.

A third invariant action is based upon the scalar

$$F^{ab}_{\ \ ik} F_{ab}^{\ \ ik} = (\partial_i L^{ab}_{\ \ k} - \partial_k L^{ab}_{\ \ i} + L^{bc}_{\ \ i} L^{a}_{\ ck} - L^{ac}_{\ \ i} L^{b}_{\ c\,k})(\partial^i L_{ab}^{\ \ k} - \partial^k L_{ab}^{\ \ i} + L_{bd}^{\ \ i} L_a^{\ dk} - L_{ad}^{\ \ i} L_b^{\ d\,k}). \tag{28}$$

It is hermitian and invariant under general coordinate transformations and local Lorentz transformations, but the squares of its time derivatives occur with opposite signs.

3. The matrix *h* and the vector *K*

The action S_L of the proposed tensor gauge fields will be invariant under local Lorentz transformations (20) if the matrix h which appears in the trace (11) transforms as

$$h' = D^{-1\dagger}\, h\, D^{-1} \tag{29}$$

where $D = D(\Lambda(x))$, $D^\dagger \beta D = \beta$ and $D \beta D^\dagger = \beta$.

The simplest choice is the hermitian matrix

$$h \ = \ i\beta\gamma^a c_a^{\ i} K_i \tag{30}$$

in which K_i is a real vector transforming as

$$K'_i(x') = \frac{\partial x^k}{\partial x'^i} K_k(x) \tag{31}$$

under general coordinate transformations. Its action S_K (4) is simpler than it looks since

$$D_i K_k - D_k K_i = \partial_i K_k - \partial_k K_i. \tag{32}$$

Under a Lorentz transformation Λ, Dirac's gamma matrices transform as a 4-vector

$$\begin{aligned} D(\Lambda)\gamma^a D^{-1}(\Lambda) &= \Lambda_b^{\ a}\, \gamma^b \\ D^{-1}(\Lambda)\gamma^a D(\Lambda) &= \Lambda^a_{\ b}\, \gamma^b \end{aligned} \tag{33}$$

where $\Lambda_b^{\ a} = \Lambda^{-1 a}_{\ \ \ \ b}$. And since the matrix $D(\Lambda)$ leaves β invariant $D^{-1\dagger}(\Lambda)\, \beta = \beta\, D(\Lambda)$ as seen earlier (22), the matrices $\beta\gamma^a$ also transform as a 4-vector

$$\begin{aligned} D^{-1\dagger}(\Lambda)\, \beta\gamma^a\, D^{-1}(\Lambda) &= \Lambda_b^{\ a}\, \beta\gamma^b \\ D^\dagger(\Lambda)\, \beta\gamma^a\, D(\Lambda) &= \Lambda^a_{\ b}\, \beta\gamma^b. \end{aligned} \tag{34}$$

So Λ changes h' to

$$h' = i\beta\gamma^a c'_a{}^i K'_i = i\beta\gamma^a \Lambda_a{}^b\, c_b{}^i\, K_i = iD^{-1\dagger}\beta\gamma^b D^{-1}\, c_b{}^i K_i = D^{-1\dagger}\, h\, D^{-1} \tag{35}$$

which satisfies (20) as does $\beta h \beta$ since

$$(\beta\, h\, \beta)' = \beta\, D^{-1\dagger} h D^{-1}\, \beta \quad = \quad D\, \beta h \beta\, D^{\dagger}. \tag{36}$$

The squares of the time derivatives $\dot{L}^{ab}{}_i$ of the gauge fields must appear with a positive sign in the action (3) if the gauge-field action is to be bounded below. They will appear with a positive sign at low temperatures if the vector boson K_i in the matrix $h = \mathrm{i}\beta\gamma^a c_a{}^i K_i$ has an average value $K_{0i} = \langle 0|K_i|0\rangle$ in the low-temperature vacuum that is timelike, $K_{0i} K^{0\,i} \simeq -m^2 < 0$. At low temperatures, the average value K_{0i} is made timelike by the second term $-\frac{1}{4}(K_i K^i + m^2)^2$ in its action S_K (4) which due to antisymmetry (32) may be written in the simpler form

$$S_K = \int \left[-\frac{1}{4}(\partial_i K_j - \partial_k K_i)(\partial^i K^j - \partial^k K^i) - \frac{1}{4}\xi^2 (K_i K^i + m^2)^2 \right] \sqrt{g}\ \mathrm{d}^4 x. \tag{37}$$

At low temperatures and presumably in the rest frame of the CMB, the vector K^i has the average vacuum value $\langle 0|K^i|0\rangle = K_0^i = m\ \delta_0^i$, and the average value of the matrix h (30) is

$$h = \mathrm{i}\beta\gamma^a c_a{}^i K_{0i} = \mathrm{i}\beta\gamma^0 c_0{}^0 K_{00} = -mI. \tag{38}$$

Then in the rest frame of the CMB and apart from the fluctuations $k^i = K^i - K_0^i$, the action S_L (24) is

$$\begin{aligned} S_L &= -\frac{1}{16\lambda^2} \int \mathrm{Tr}[(\boldsymbol{R}_{ik} \cdot \boldsymbol{\sigma} I + \mathrm{i}\boldsymbol{B}_{ik} \cdot \boldsymbol{\sigma}\gamma^5)(\boldsymbol{R}^{ik} \cdot \boldsymbol{\sigma} I - \mathrm{i}\boldsymbol{B}^{ik} \cdot \boldsymbol{\sigma}\gamma^5)] \sqrt{g}\ \mathrm{d}^4 x \\ &= -\frac{1}{4\lambda^2} \int (\boldsymbol{R}_{ik} \cdot \boldsymbol{R}^{ik} + \boldsymbol{B}_{ik} \cdot \boldsymbol{B}^{ik}) \sqrt{g}\ \mathrm{d}^4 x \end{aligned} \tag{39}$$

in which the squares of the time derivatives $(\dot{\boldsymbol{r}}_{\boldsymbol{k}})^2$ and $(\dot{\boldsymbol{b}}_{\boldsymbol{k}})^2$ appear with positive signs as promised. So the action S_L is bounded below in the rest frame of the CMB.

At low temperatures, the vector boson K_i fluctuates about its average value, $K_i(x) = K_{0i} + k_i(x)$. The fluctuations $k_i(x)$ are those of a massless vector field with $k_0(x) = 0$ as discussed in appendix C. The six gauge fields $L^{ab}{}_i$ remain massless despite the nonzero average value K_0^i of the vector boson K^i.

The ground state $|0, \boldsymbol{v}\rangle$ of a Lorentz frame moving at velocity $\boldsymbol{v}$ relative to the CMB is related to the ground state of the CMB by a unitary Lorentz transformation $|0, \boldsymbol{v}\rangle = U_{\boldsymbol{v}}|0\rangle$ that represents matched (62) Lorentz and general-coordinate transformations. The action is invariant under Lorentz and general-coordinate transformations

$$U_{\boldsymbol{v}}^{-1}\, S_L\, U_{\boldsymbol{v}} = S_L. \tag{40}$$

So the average value of the action in the state $|0, \boldsymbol{v}\rangle$ is the same as in the state $|0\rangle$ in which the CMB is at rest

$$\langle 0, \boldsymbol{v}|S_L|0, \boldsymbol{v}\rangle = \langle 0|U_{\boldsymbol{v}}^{-1} S_L\, U_{\boldsymbol{v}}|0\rangle = \langle 0|S_L|0\rangle. \tag{41}$$

Thus the action is bounded below in all Lorentz frames.

The matrix h takes a simpler form in two-component notation as discussed in appendix D.

It may be useful here to distinguish different kinds of symmetry. One kind is an exact symmetry of the action and of the vacuum, like that of the group $SU_c(3)$ of QCD.

A second kind is a symmetry of the action but not of the vacuum, like that of $\mathrm{SU}_\ell(2) \otimes \mathrm{U}(1)_Y$ in which the average value of a component of the Higgs field in the low-temperature vacuum makes the W and Z bosons massive. Their masses make their interactions of short range and therefore weak.

A third kind is symmetries of the action that are intrinsically reduced by the ground state of the actual universe. As described in section 5, the average values of the tetrads $c^a_{0i}(x) = \mathrm{Tr}(\rho\, c^a_{\ i}(x))$ in the ground state of the late, low-temperature universe reduce the two independent symmetries of general coordinate and local Lorentz invariance to a single exact symmetry of the vacuum—local Lorentz invariance. Every local Lorentz transformation $\Lambda^a{}_b(x)$ must be accompanied by a specific general coordinate transformation (66)

$$\frac{\partial x'^i}{\partial x^k} = c_a{}^i(x)\, \Lambda^a{}_b(x)\, c^b{}_k(x) \tag{42}$$

in order to preserve the average values of the tetrads.

In the theory sketched in this paper, the average value of the vector boson K_i in the present low-temperature universe and in the rest frame of the CMB, $\langle 0|K_i|0\rangle = m\ \delta^0_i$, gives the action S_L (3, 24) of the six gauge fields $L^{ab}_{\ \ i}$ the approximate and simpler form (39). The six gauge fields $L^{ab}_{\ \ i}$ remain massless, and local Lorentz invariance remains exact—apart from $\langle 0|K_i|0\rangle$, the CMB, and the matter and energy of the actual universe.

4. Dirac action

The explicitly hermitian Dirac action is

$$S_D = -\frac{1}{2}\int [\bar{\psi}\ \gamma^a c_a{}^i D_i\psi + (\bar{\psi}\ \gamma^a c_a{}^i D_i\psi)^\dagger]\sqrt{g}\ \mathrm{d}^4x \tag{43}$$

in which the covariant derivative $D_i\psi$ is

$$D_i\psi = \left(\partial_i + \tfrac{1}{8}\, L^{ab}{}_i[\gamma_a, \gamma_b]\right)\psi. \tag{44}$$

To avoid clutter, I am using a single Dirac field ψ and am suppressing the gauge bosons A^α_i of $SU_c(3)\times SU_\ell(2)\times U(1)$. To include them, one would replace the single Dirac field ψ by a vector Ψ whose components Ψ_α would be the 6 leptons and 18 quark fields. One would add the 12 gauge bosons A^α_i of $SU_c(3)\times SU_\ell(2)\times U(1)$ and their actions. Then the covariant derivative of the 96-component Dirac field Ψ would be

$$D_i\Psi = \left(\partial_i + \tfrac{1}{8}\, L^{ab}{}_i[\gamma_a, \gamma_b] + A^\alpha_i t^\alpha\right)\Psi \tag{45}$$

in which the t^α's are the generators of the Lie algebras of $SU_c(3)$, $SU_\ell(2)$ and $U(1)_Y$.

The simplest choice for $\bar{\psi}$ is Dirac's choice $\bar{\psi} = \psi^\dagger\beta$

$$\mathcal{L}_D = -\psi^\dagger\beta\ c_b{}^i\gamma^b D_i\psi \tag{46}$$

in which the real 4×4 hermitian symmetric matrix $\beta = i\gamma^0$ obeys $D^\dagger(\Lambda)\,\beta\, D(\Lambda) = \beta$. The resulting Dirac action (8) is then invariant under local Lorentz and general coordinate transformations.

Under a local Lorentz transformation $D = D(\Lambda(x))$, the Dirac field ψ, its covariant derivative $D_i\psi$ and the tetrads $c_a{}^i$ transform as

$$\begin{aligned} \psi' &= D\,\psi \\ \bar{\psi}' &= (\psi^\dagger \beta)' = \psi^\dagger D^\dagger \beta = \psi^\dagger \beta\, D^{-1} \\ (D_i \psi)' &= D\, D_i \psi \\ ({c_a}^i)' &= {\Lambda_a}^b\, {c_b}^i \end{aligned} \tag{47}$$

while $D\,\gamma^a\,D^{-1} = {\Lambda_b}^a\,\gamma^b = {\Lambda^{-1a}}_b \gamma^b$. Thus the action density (8) is invariant under local Lorentz transformations

$$\mathcal{L}_D' = (\psi^\dagger \beta\,\gamma^a\,{c_a}^i D_i \psi)' = \psi^\dagger D^\dagger\,\beta\,\gamma^a D\,{c_c}^i {\Lambda_a}^c D_i \psi = \psi^\dagger \beta\,\gamma^b\,{\Lambda^a}_b {\Lambda^{-1c}}_a\,{c_c}^i D_i \psi = \psi^\dagger \beta\,\gamma^b\,{c_b}^i D_i \psi = \mathcal{L}_D \tag{48}$$

as well as under general coordinate transformations.

The explicitly hermitian action density is

$$\mathcal{L}_{Dh} = -\psi^\dagger \beta {c_a}^i \gamma^a \partial_i \psi - \tfrac{1}{2}\psi^\dagger\,\beta\,\gamma^a (\partial_i {c_a}^i)\,\psi + \psi^\dagger \tfrac{1}{2}({c_0}^i \boldsymbol{r}_i \cdot \boldsymbol{\sigma}\,I - {c_s}^i {r_i}^s \gamma^5)\psi - \tfrac{1}{2}\,\epsilon_{jsk} {c_j}^i {b_i}^s \psi^\dagger \sigma^k \psi. \tag{49}$$

as shown in appendix E.

Although the current that generates the rotational field r_i^s is

$$j_r^{is} = \tfrac{1}{2}\psi^\dagger ({c_0}^i \sigma^s - {c_s}^i \gamma^5)\psi, \tag{50}$$

the current that generates the boost field b_i^s has no time component

$$j_b^{is} = -\tfrac{1}{2}\epsilon_{jsk} {c_j}^i \psi^\dagger \sigma^k \psi. \tag{51}$$

So unless the spatial tetrads are nondiagonal so that ${c_j}^0 \neq 0$ for $j = 1$, 2 or 3, the time components of the boost bosons ${b_0}^s$ do not occur in the Dirac action, and do not generate Coulomb potentials.

These comments apply also to the gauge fields of groups larger than the Lorentz group: unless the spatial tetrads are nondiagonal, ${c_a}^0 \neq 0$ for $a > 0$, the time components of the gauge bosons of the generators of the noncompact directions do not appear in the Dirac action and do not generate Coulomb potentials.

In the static limit, the exchange of the three massless tensor gauge fields $\boldsymbol{r}_i$ that gauge rotations would imply that two macroscopic bodies of F and F' fermions separated by a distance r would contribute to the energy a static Coulomb potential

$$K_L(r) = \frac{3\,FF'f^2}{4\pi r}. \tag{52}$$

This potential is positive and repulsive (between fermions and between antifermions). It violates the weak equivalence principle because it depends upon the number F of fermions as $F = 3B + L$ and not upon their masses.

The potential $K_L(r)$ changes Newton's potential to

$$K_{NL}(r) = -G\,\frac{mm'}{r}(1 + \alpha) \tag{53}$$

in which the repulsive coupling strength α is

$$\alpha = -\frac{3FF'\lambda^2}{4\pi Gmm'} = -\frac{3FF'm_{\rm P}^2\lambda^2}{4\pi mm'}. \tag{54}$$

This force would increase the need for dark matter and decrease the need for dark energy. Couplings $\alpha \sim 1$ are of gravitational strength.

Experiments [11–34] put no upper limits on the masses of tensor gauge fields and no lower limits on their coupling λ.

5. Intrinsic reduction of symmetry

When quantizing a gauge theory, one picks a gauge. For general relativity in flat space, the usual gauge choice is to set the average value of the metric $g_{ik}(x)$ equal to the Minkowski metric η

$$\langle 0|g_{ik}(x)|0\rangle = \eta_{ik} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{55}$$

The average vacuum value of the metric is then quadratic in the average values of Cartan's tetrads

$$\langle 0|g_{ik}(x)|0\rangle = \langle 0|c^a_{\ i}(x)\,\eta_{ab}\,c^b_{\ k}(x)|0\rangle = \eta_{ik}. \tag{56}$$

A further gauge choice of Lorentz frame sets the average vacuum values of the tetrads equal to Kronecker deltas

$$\langle 0|c^a_{\ i}(x)|0\rangle = \delta^a_{\ i}. \tag{57}$$

The independent symmetry transformations of general coordinate invariance (1)

$$c'^b_{\ \ i}(x') = \frac{\partial x^k}{\partial x'^i}\,c^b_{\ k}(x) \tag{58}$$

and local Lorentz invariance (2)

$$c'^a_{\ \ i}(x) = \Lambda^a_{\ b}(x)\,c^b_{\ i}(x) \tag{59}$$

map a tetrad $c^b_{\ k}(x)$ to

$$c'^a_{\ \ i}(x') = \Lambda^a_{\ b}(x)\,\frac{\partial x^k}{\partial x'^i}\,c^b_{\ k}(x) \tag{60}$$

So if the average vacuum values (57) of the tetrads $\langle 0|c^a_{\ i}(x)|0\rangle = \delta^a_{\ i}$ are to be invariant, then

$$\Lambda^a_{\ b}(x)\,\frac{\partial x^k}{\partial x'^i}\,\delta^b_{\ k} = \Lambda^a_{\ k}(x)\,\frac{\partial x^k}{\partial x'^i} = \delta^a_{\ i}. \tag{61}$$

By multiplying this last equation by $\partial x'^i/\partial x^\ell$, we see that the tetrad values $\langle 0|c^a_{\ i}(x)|0\rangle = \delta^a_{\ i}$ will be unchanged only if the general coordinate transformation $x \to x'$ and the local Lorentz transformation $\Lambda^a_{\ \ell}(x)$

$$\frac{\partial x'^a}{\partial x^\ell} = \Lambda^a_{\ \ell}(x) \quad \Longleftrightarrow \quad \frac{\partial x^k}{\partial x'^i} = \Lambda^{-1k}_{\ \ \ \ i}(x) \tag{62}$$

are the same. In that case, we have

$$\Lambda^a_{\ k}\,\frac{\partial x^k}{\partial x'^i} = \Lambda^a_{\ k}\,\Lambda^{-1k}_{\ \ \ \ i}(x) = \delta^a_{\ i} \tag{63}$$

which maintains the average values of the tetrads in the vacuum

$$\langle 0|c'^a_{\ \ i}(x')|0\rangle = \Lambda^a_{\ b}(x)\,\frac{\partial x^k}{\partial x'^i}\,\langle 0|c^b_{\ k}(x)|0\rangle = \Lambda^a_{\ b}(x)\,\frac{\partial x^k}{\partial x'^i}\,\delta^b_{\ k} = \Lambda^a_{\ k}(x)\,\Lambda^{-1k}_{\ \ \ \ i}(x) = \delta^a_{\ i}. \tag{64}$$

In a universe described by a density operator ρ, the average values of the tetrads are traces

$$c^a_{\ i}(x) = \mathrm{Tr}(\rho\; c^a_{\ i}(x)). \tag{65}$$

By using the tetrad identity $c_a{}^k c^a{}_i = \delta^k_i$, one may show that the joint transformations (60) preserve the average values (65) of the tetrads if the general coordinate transformation $x \to x'$ is related to the local Lorentz transformation $\Lambda(x)$ by two tetrads

$$\frac{\partial x'^i}{\partial x^k} = c_a{}^i(x)\,\Lambda^a{}_b(x)\,c^b{}_k(x). \tag{66}$$

The nonzero average values of the tetrads reduce the two symmetries of the action to a single symmetry of the ground state of the Universe: local Lorentz invariance. This reduction of symmetry is intrinsic rather than spontaneous because tetrads intrinsically have nonzero average values $\mathrm{Tr}(\rho\, c^a_i(x))$.

The ideas of this section are independent of the existence of the tensor gauge fields proposed in this paper.

6. Is the gauge group U(2, 2)?

So far we have been assuming that the gauge group of the Dirac field is the Lorentz group which acts on the Dirac field as the direct sum

$$D(\Lambda) = D^{(1/2,0)}(\Lambda) \oplus D^{(0,1/2)}(a). \tag{67}$$

Is this a hint of a larger symmetry? These 4×4 matrices $D(\Lambda) = D(\Lambda(x))$ leave $\beta = i\gamma^0$ invariant

$$D^\dagger(\Lambda)\,\beta\,D(\Lambda) = \beta \quad \text{and} \quad D(\Lambda)\beta D^\dagger(\Lambda) = \beta. \tag{68}$$

as noted earlier (22) and transform Dirac's gamma matrices as a 4-vector

$$D^\dagger(\Lambda)\,\beta\gamma^a\,D(\Lambda) = \Lambda^a{}_b\,\beta\gamma^b \tag{69}$$

as noted earlier (33 and 34).

The gauge group of the Dirac field may be the group of all 4×4 complex matrices U that leave β invariant

$$D^\dagger(U)\,\beta\,D(U) = \beta. \tag{70}$$

This group has 16 generators and is known as U(2,2) ([35]) as one may see by rotating all four of the matrices of this equation of β invariance (70) from the $\beta = \mathrm{i}\gamma^0$ direction to the γ^5 direction. This rotation shows that the group must leave γ^5 invariant

$$D^\dagger(U')\,\gamma^5\,D(U') = \gamma^5 \tag{71}$$

which is the defining equation of U(2,2).

The 4×4 direct-sum matrices $D(\Lambda)$ (67) leave β invariant (68) and so form a subgroup of U(2, 2).

To implement U(2, 2) gauge symmetry, we'll need to extend the 6 generators $\mathcal{J}^{ab} = -\frac{1}{4}i[\gamma^a, \gamma^b]$ and 6 gauge fields L^{ab}_i to 16 generators G^A and 16 gauge fields L^A_i so that

$$\begin{aligned}
D^\dagger(U)\beta G^A D(U) &= \mathscr{D}(U)^A{}_B\,\beta G^B\\
D^{-1}(U) G^A D(U) &= \mathscr{D}(U)^A{}_B\,G^B\\
\psi' &= D(U)\,\psi\\
D_i\psi &= \left(\partial_i + \frac{1}{2}\mathrm{i}\,L^A_i G_A\right)\psi\\
(D_i\psi)' &= D(U) D_i\psi
\end{aligned} \tag{72}$$

in which $\mathscr{D}(U)^A{}_B$ is the 16×16 matrix that represents U in the adjoint representation of $U(2, 2)$. The Dirac action will be invariant under these local U(2, 2) transformations (72) if Cartan's tetrads are also extended from four 4-vectors to 16 4-vectors transforming as

$$e'^A{}_i = U^A_B\, e^B_i \tag{73}$$

The metric g_{ik} must be invariant under U(2, 2)

$$g_{ik} = e^A{}_i e_{Ak} = e'^A{}_i\, e'{}_{Ak} = g'{}_{ik}. \tag{74}$$

We have seen (73) that $e'^A{}_i = U^A_B\, e^B_i$, but we can choose how e_{Ai} transforms. If we choose

$$e'{}_{A\,k} = F_A{}^C\, e_{Ck} \tag{75}$$

then to keep

$$g_{ik} = e^A{}_i e_{fk} = e'^f{}_i\, e'{}_{fk} = e^h_i\, F^A{}_C e_{Bk}\, U^B_A \tag{76}$$

we need

$$U^A_B\, F_A{}^C = \delta^C_B \quad \text{or} \quad (F^{-1})_B{}^A = U^A_B. \tag{77}$$

for then we'd have

$$g_{ik} = e^A{}_i e_{Ak} = e'^A{}_i\, e'{}_{Ak} = U^A_B\, e^B_i\, F_A{}^C\, e_{Ck} = e^B_i\, \delta^C_B\, e_{Ck} = e^C{}_i e_{Ck} = g_{ik}. \tag{78}$$

The real Lie algebras su(2,2) and sl(4,R) are not isomorphic, but over the complex numbers they both belong to A_3 in the Cartan-Weyl classification, so their complexifications are isomorphic [36]. It therefore may make sense to consider the possibility that GL(4,R) or GL(4,C) is the gauge group of the Dirac field.

7. Cartan's first equation of structure

If we add all invariant terms of dimension (mass)^4 or less to the action (which is by no means required in a non-renormalizable theory of gravity), then the covariant derivatives (10) of the tetrads

$$D_i c^a_k = \partial_i c^a_k + \; L_i{}^a{}_b c^b_k - \Gamma^\ell{}_{ki}\, c^a_\ell \tag{79}$$

would appear in the action squared and contracted

$$S_C = -M^2 \int D_i c^a_k D^i c_a{}^k \sqrt{g}\; \mathrm{d}^4x \tag{80}$$

the coefficient M^2 being required because tetrads are dimensionless.

If the mass M in the action S_C is huge, say of the order of the Planck mass M_P, then the equation of motion of the tensor gauge fields would be approximately the condition that S_C be stationary

$$\begin{aligned} \delta S_C &= -M^2 \int [(\delta D_i c^a_k) D^i c_a{}^k + D_i c^a_k \delta D^i c_a{}^k] \sqrt{g}\; \mathrm{d}^4x \\ &= -2M^2 \int (\delta L_i{}^a{}_b) c^b_k D^i c_a{}^k \sqrt{g}\; \mathrm{d}^4x = 0 \end{aligned} \tag{81}$$

because the other terms in the action that contain the fields $L_i{}^a{}_b$ —namely the action terms S_D, S_L and S_K (4, 8 and 11)—lack the huge coefficient M^2.

Thus in the limit $M^2 \to \infty$, the equation of motion of the tensor gauge fields is

$$0 = D_i c^a_k = \partial_i c^a_k + L_i{}^a{}_b c^b_k - \Gamma^\ell{}_{ki}\, c^a_\ell \tag{82}$$

which is Cartan's first equation of structure usually derived from the tetrad postulate or from the assumption that the torsion vanishes. Multiplying it by c^{ck}, we find that in the $M^2 \to \infty$ limit, the tensor gauge fields approach the spin connection

$$L_i{}^{ac} \simeq \Gamma^{\ell}{}_{ki}\, c^a{}_{\ell} c^{ck} + c^a{}_k\, \partial_i c^{ck} = \omega_i{}^{ac}. \tag{83}$$

8. Conclusions

The action of general relativity with fermions has two independent symmetries: invariance under general coordinate transformations and invariance under local Lorentz transformations. The action of local Lorentz transformations on Dirac and Lorentz indices is similar to the action of noncompact internal-symmetry transformations on Lie-group indices.

The internal-symmetry character of local Lorentz invariance suggests that it might be implemented not by the spin connection but by tensor gauge fields $L^{ab}{}_i$ with their own Yang–Mills action. But because the Lorentz group is noncompact, their Yang–Mills action must be modified by a neutral vector field $K_i(x)$ whose average value at low temperatures is timelike. This vector boson is massless at low temperatures. The vector gauge fields $L^{ab}{}_i$ are massless at all temperatures.

The particles of the neutral, gravitationally interacting and massless fields $L^{ab}{}_i$ and K_i would contribute to the hot dark matter of the Universe. The massive particles radiated by the vector field K at high temperatures would contribute at lower temperatures to cold dark matter.

The nonzero average values of the tetrads reduce the spacetime symmetries of the vacuum to local Lorentz invariance which can be extended to local U(2,2) invariance.

If the contracted squares of the covariant derivatives of the tetrads multiplied by the square of a mass M are added to the action, then in the limit $M^2 \to \infty$, the equation of motion of the tensor gauge fields is the vanishing of the covariant derivatives of the tetrads, which is Cartan's first equation of structure. In the same limit, the tensor gauge fields approach the spin connection.

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Data availability statement

No new data were created or analyzed in this study.

Appendix A. The matrix $D(\Lambda)$

The matrix $D(\Lambda(\boldsymbol{\theta}, \boldsymbol{\lambda}))$ is

$$D(\Lambda(\boldsymbol{\theta}, \boldsymbol{\lambda})) = \begin{pmatrix} D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) & 0 \\ 0 & D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} e^{-z\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{z^*\cdot\boldsymbol{\sigma}} \end{pmatrix} \tag{A1}$$

in which $z = \frac{1}{2}(\boldsymbol{\lambda} + i\boldsymbol{\theta})$ ([37]). So

$$D\beta = \begin{pmatrix} e^{-z\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{z^*\cdot\boldsymbol{\sigma}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-z\cdot\boldsymbol{\sigma}} \\ e^{z^*\cdot\boldsymbol{\sigma}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{z^*\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{-z\cdot\boldsymbol{\sigma}} \end{pmatrix} = \beta D^{\dagger -1} \tag{A2}$$

and

$$\beta D^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{z\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{-z^*\cdot\boldsymbol{\sigma}} \end{pmatrix} = \begin{pmatrix} 0 & e^{-z\cdot\boldsymbol{\sigma}} \\ e^{z\cdot\boldsymbol{\sigma}} & 0 \end{pmatrix} = \begin{pmatrix} e^{-z^*\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{z\cdot\boldsymbol{\sigma}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D^\dagger\beta \tag{A3}$$

as noted earlier (22).

Appendix B. Explicit form of S_L in frame of earth

The Solar System moves at $v = 368 \pm 2$ km s^{-1} relative to the CMB. So in the Lorentz frame of the Earth, the average value of K^i is $K^i_{0v} = \langle 0, v|K^i|0, v\rangle = m(1, -\boldsymbol{v})/\sqrt{1 - v^2}$. The average value of the matrix $h = \mathrm{i}\beta\gamma^a c_a{}^i K_i$ is then

$$\langle 0, E|h|0, E\rangle = m\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5 - I}{\sqrt{1 - v^2}}. \tag{B1}$$

Since $\beta\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5\beta = -\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5$, the average value of $\beta h\beta$ is

$$\langle 0, E|\beta h\beta|0, E\rangle = -m\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5 + I}{\sqrt{1 - v^2}}. \tag{B2}$$

So at low temperatures and apart from the fluctuations $k_i(x)$, the action S_L (24) is

$$S_L = -\frac{1}{16m^2\lambda^2}\int \mathrm{Tr}[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I + i\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5)h(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I - i\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)\beta h\beta]\sqrt{g}\;\mathrm{d}^4x$$
$$= \frac{1}{16\lambda^2}\int \mathrm{Tr}\left[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I + i\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5 - I}{1 - v^2}(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I - i\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)(\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5 + I)\right]\sqrt{g}\;\mathrm{d}^4x. \tag{B3}$$

Since $v \simeq 10^{-3}$, it is useful to separate terms according to the number of powers of v

$$S_L = -\frac{1}{16\lambda^2}\int \mathrm{Tr}[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I + i\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5)(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I - i\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)]\frac{\sqrt{g}\;\mathrm{d}^4x}{1 - v^2} \tag{B4}$$

$$-\frac{1}{16\lambda^2}\int \mathrm{Tr}[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I + i\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5)(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I - i\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5]\sqrt{g}\;\mathrm{d}^4x \tag{B5}$$

$$+\frac{1}{16\lambda^2}\int \mathrm{Tr}[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I + i\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5)\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I - i\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)]\sqrt{g}\;\mathrm{d}^4x \tag{B6}$$

$$+\frac{1}{16\lambda^2}\int \mathrm{Tr}[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I + i\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5)\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I - i\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5] \tag{B7}$$

in which the first integral contains terms of order zero and two in v.

Terms with an odd number of γ^5's cancel. The first term (B4) is thus

$$I_1 = -\frac{1}{4\lambda^2}\int (\boldsymbol{R}_{ik}\cdot\boldsymbol{R}^{ik} + \boldsymbol{B}_{ik}\cdot\boldsymbol{B}^{ik})\frac{\sqrt{g}\ \mathrm{d}^4x}{1-v^2}. \tag{B8}$$

The second (B5) and third (B6) integrals involve a commutator

$$\begin{aligned} I_2 + I_3 &= \frac{1}{16\lambda^2}\int \mathrm{Tr}\{[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I - i\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5),(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I + i\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)]\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5\}\sqrt{g}\ \mathrm{d}^4x \\ &= \frac{i}{8\lambda^2}\int \mathrm{Tr}\{[\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I, \boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}]\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\}\sqrt{g}\ \mathrm{d}^4x = -\frac{1}{\lambda^2}\int \boldsymbol{R}_{ik}\times\boldsymbol{B}^{ik}\cdot\boldsymbol{v}\sqrt{g}\ \mathrm{d}^4x. \end{aligned} \tag{B9}$$

The fourth term (B7) is quadratic in v

$$\begin{aligned} I_4 &= \frac{1}{16\lambda^2}\int \mathrm{Tr}[(\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}I + \mathrm{i}\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma}\gamma^5)\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5(\boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma}I - \mathrm{i}\boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma}\gamma^5)\boldsymbol{\sigma}\cdot\boldsymbol{v}\gamma^5]\sqrt{g}\ \mathrm{d}^4x \\ &= \frac{1}{16\lambda^2}\int \mathrm{Tr}[\boldsymbol{R}_{ik}\cdot\boldsymbol{\sigma}\ \boldsymbol{\sigma}\cdot\boldsymbol{v}\ \boldsymbol{R}^{ik}\cdot\boldsymbol{\sigma\sigma}\cdot\boldsymbol{v}\boldsymbol{B}_{ik}\cdot\boldsymbol{\sigma\sigma}\cdot\boldsymbol{v}\ \boldsymbol{B}^{ik}\cdot\boldsymbol{\sigma\sigma}\cdot\boldsymbol{v}]\sqrt{g}\ \mathrm{d}^4x. \\ &= \frac{1}{4\lambda^2}\int [2\boldsymbol{R}_{ik}\cdot\boldsymbol{v}\boldsymbol{R}^{ik}\cdot\boldsymbol{v} - \boldsymbol{R}_{ik}\cdot\boldsymbol{R}^{ik}\ \boldsymbol{v}\cdot\boldsymbol{v}2\boldsymbol{B}_{ik}\cdot\boldsymbol{v}\boldsymbol{B}^{ik}\cdot\boldsymbol{v} - \boldsymbol{B}_{ik}\cdot\boldsymbol{B}^{ik}\ \boldsymbol{v}\cdot\boldsymbol{v}]\sqrt{g}\ \mathrm{d}^4x. \end{aligned} \tag{B10}$$

The action S_L in the frame of the Earth is the sum

$$S_L = I_1 + I_2 + I_3 + I_4. \tag{B11}$$

Appendix C. The vector boson K^i

At high temperatures, the Lagrange density

$$L_K = -\tfrac{1}{4}(\partial_i K_j - \partial_j K_i)(\partial^i K^j - \partial^j K^i) - \tfrac{1}{4}\xi^2 m^4 - \tfrac{1}{2}\xi^2 m^2 K_i K^i - \tfrac{1}{4}\xi^2 K_i K^i K_j K^j \tag{C1}$$

of the action S_K (37) describes a spin-one vector boson K^i that of mass ξm. The particles of the boson K_i would have contributed to the cold dark matter of the universe at temperatures less than ξm.

At the low temperatures of the present universe, the same Lagrange density describes a vector boson K^i whose average value $\langle 0|K^i(x)|0\rangle$ is timelike; in some Lorentz frame, presumably the rest frame of the CMB, the vector $K^i(x) = m\delta^i_0 + k^i(x)$ has average values $\langle 0|K^0(x)|0\rangle = m$ and $\langle 0|k^i(x)|0\rangle = 0$ which make $\langle 0|K^i(x)|0\rangle$ timelike. Its small fluctuations $k^i(x)$ are those of a massless vector $\boldsymbol{k}(x)$ described by the Lagrange density

$$L_k = -\tfrac{1}{4}(\partial_i k_j - \partial_j k_i)(\partial^i k^j - \partial^j k^i) - \tfrac{1}{4}\xi^2 m^4 - \tfrac{1}{2}\xi^2 m^2(\boldsymbol{k}^2 - (m + k^0)^2) - \tfrac{1}{4}\xi^2(\boldsymbol{k}^2 - (m + k^0)^2)^2 \tag{C2}$$

or

$$\begin{aligned} L_k = &- \tfrac{1}{4}(\partial_i k_j - \partial_j k_i)(\partial^i k^j - \partial^j k^i) - \tfrac{1}{4}\xi^2 m^4 - \tfrac{1}{2}\xi^2 m^2(\boldsymbol{k}^2 - m^2 - 2mk^0 - k^{02}) \\ &- \tfrac{1}{4}\xi^2(\boldsymbol{k}^2)^2 + \tfrac{1}{2}\xi^2\boldsymbol{k}^2(m^2 + 2mk^0 + k^{02}) - \tfrac{1}{4}\xi^2(m^4 + 4m^3k^0 + 6m^2k^{02} + 4mk^{03} + k^{04}). \end{aligned} \tag{C3}$$

These massless particles would contribute hot dark matter to the present universe.

Combining terms, we get

$$L_k = -\tfrac{1}{4}(\partial_i k_j - \partial_j k_i)(\partial^i k^j - \partial^j k^i) - \tfrac{1}{4}\xi^2 m^4 + \tfrac{1}{2}\xi^2 m^4 - \tfrac{1}{4}\xi^2 m^4 + \xi^2 m^3 k^0 - m^3 \xi^2 k^0$$
$$- \tfrac{1}{2}\xi^2 m^2 \boldsymbol{k}^2 + \tfrac{1}{2}\xi^2 m^2 \boldsymbol{k}^2 + \tfrac{1}{2}\xi^2 m^2 k^{02} - \tfrac{3}{2}\xi^2 m^2 k^{02} + m\xi^2 \boldsymbol{k}^2 k^0 - m\xi^2 k^{03} - \tfrac{1}{4}\xi^2 (\boldsymbol{k}^2)^2 + \tfrac{1}{2}\xi^2 \boldsymbol{k}^2 k^{02} - \tfrac{1}{4}\xi^2 k^{04} \tag{C4}$$

or

$$L_k = -\tfrac{1}{4}(\partial_i k_j - \partial_j k_i)(\partial^i k^j - \partial^j k^i) - \xi^2 m^2 k^{02} + \xi^2 m \boldsymbol{k}^2 k^0 - \xi^2 m k^{03} - \tfrac{1}{4}\xi^2 (\boldsymbol{k}^2)^2 + \tfrac{1}{2}\xi^2 \boldsymbol{k}^2 k^{02} - \tfrac{1}{4}\xi^2 k^{04}. \tag{C5}$$

The quadratic part of L_k is

$$L_{k2} = -\frac{1}{4}(\partial_i k_j - \partial_j k_i)(\partial^i k^j - \partial^j k^i) - \xi^2 m^2 k^{02}. \tag{C6}$$

The linear Euler–Lagrange equations for $k^i(x)$ are

$$\partial_j(\partial^j k^i - \partial^i k^j) = -2\xi^2 m^2\, k^0\, \delta^i_0 \tag{C7}$$

or

$$\triangle k^0 + \nabla \cdot \dot{\boldsymbol{k}} = -2\xi^2 m^2 k^0 \tag{C8}$$

and

$$-\ddot{\boldsymbol{k}} + \triangle\, \boldsymbol{k} - \nabla(\dot{k}^0 + \nabla \cdot \boldsymbol{k}) = 0. \tag{C9}$$

One solution is $-\triangle k^0 = 2m^2 k^0$, $\nabla \cdot \boldsymbol{k} = 0$, $\dot{k}^0 = 0$, and $\partial_j \partial^j \boldsymbol{k} = 0$.

Appendix D. Two-component formalism

Dirac's formalism is economical, but the two-component formalism is better suited to a discussion of a new interpretation of the matrix h.

Since the Dirac–Lorentz matrix (A1) is block diagonal

$$D(\boldsymbol{\theta}, \boldsymbol{\lambda})\begin{pmatrix} D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) & 0 \\ 0 & D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{\boldsymbol{z}^*\cdot\boldsymbol{\sigma}} \end{pmatrix} \equiv \begin{pmatrix} D_\ell & 0 \\ 0 & D_r \end{pmatrix}, \tag{D1}$$

the matrix h

$$h = \begin{pmatrix} h_\ell & 0 \\ 0 & h_r \end{pmatrix} \tag{D2}$$

and its transformation law $h' = D^{-1\dagger}\, h\, D^{-1}$ (29) are block diagonal

$$\begin{pmatrix} h'_\ell & 0 \\ 0 & h'_r \end{pmatrix} = \begin{pmatrix} D_\ell^{-1\dagger} h_\ell D_\ell^{-1} & 0 \\ 0 & D_r^{-1\dagger} h_r D_r^{-1} \end{pmatrix}. \tag{D3}$$

The matrix $\beta h \beta$ is just h with left and right interchanged

$$\beta h \beta = \begin{pmatrix} h_r & 0 \\ 0 & h_\ell \end{pmatrix}. \tag{D4}$$

The transformation law $\psi' = D\, \psi$ of a Dirac field

$$\psi = \begin{pmatrix} \ell \\ r \end{pmatrix} \tag{D5}$$

is

$$\psi' = \begin{pmatrix} \ell' \\ r' \end{pmatrix} = \begin{pmatrix} D_\ell\ \ell \\ D_r\ r \end{pmatrix}. \tag{D6}$$

So the combinations $\ell^\dagger h_\ell \ell$ and $r^\dagger h_r r$ are invariant

$$\begin{aligned} (\ell^\dagger h_\ell \ell)' &= \ell^\dagger D_\ell^\dagger D_\ell^{-1\dagger} h_\ell D_\ell^{-1} D_\ell \ell = \ell^\dagger h_\ell \ell \\ (r^\dagger h_r r)' &= r^\dagger D_r^\dagger D_r^{-1\dagger} h_r D_r^{-1} D_r r = r^\dagger h_r r \end{aligned} \tag{D7}$$

much as contracted tensors are invariant

$$(X^i g_{ik} Y^k)' = X'^i g'_{ik} Y'^k \quad = \quad X^i g_{ik} Y^k. \tag{D8}$$

We now see that the 2×2 matrices h_ℓ and h_r do for spinor indices what g_{ik} does for tensor indices.

Now if $D_\ell = e^{-z\cdot\sigma/2}$, then $D_r = e^{z^*\cdot\sigma/2}$, and so $D_r = D_\ell^{\dagger -1}$. So we can set $h_r = h_\ell^{-1\dagger}$.

Thus in two-component notation, the gauge-field action S_L (24) is

$$\begin{aligned} S_L = -\frac{1}{16m^2} \int \{ &\mathrm{Tr}[(\boldsymbol{R}_{ik} + \mathrm{i}\boldsymbol{B}_{ik}) \cdot \boldsymbol{\sigma} h_\ell (\boldsymbol{R}^{ik} - i\boldsymbol{B}^{ik}) \cdot \boldsymbol{\sigma} h_\ell^{-1\dagger}] \\ &+ \mathrm{Tr}[(\boldsymbol{R}_{ik} + i\boldsymbol{B}_{ik}) \cdot \boldsymbol{\sigma} h_\ell^{-1\dagger} (\boldsymbol{R}^{ik} + i\boldsymbol{B}^{ik}) \cdot \boldsymbol{\sigma} h_\ell]\} \sqrt{g}\ \mathrm{d}^4 x. \end{aligned} \tag{D9}$$

Since $h'_\ell = D_\ell^{-1\dagger} h_\ell D_\ell^{-1}$, we may choose h_ℓ to be hermitian $h_\ell^\dagger = h_\ell$, which implies that the diagonal form of h_ℓ is just two real numbers. The most general 2×2 hermitian matrix is a linear combination of the Pauli matices $\boldsymbol{\sigma}$ and the identity matrix I. Under a Lorentz transformation Λ, the 4-vector $s_\ell^a \equiv (-I, \boldsymbol{\sigma})$ transforms as

$$D_\ell^\dagger(\Lambda)\, s_\ell^a\, D_\ell(\Lambda) = \Lambda^a{}_b\, s_\ell^b \tag{D10}$$

while the 4-vector $s_r^a \equiv (I, \boldsymbol{\sigma})$ transforms as

$$D_r^\dagger(\Lambda)\, s_r^a\, D_r(\Lambda) = \Lambda^a{}_b\, s_r^b. \tag{D11}$$

The 4×4 matrix h (30) is

$$h \;=\; i\beta\gamma^a c_a{}^i K_i = i\beta\gamma^a K_a = \begin{pmatrix} s_\ell^a K_a & 0 \\ 0 & s_r^a K_a \end{pmatrix}. \tag{D12}$$

Under a local Lorentz transformation $\Lambda = \Lambda(x)$, the vector field $K_a(x)$ goes to

$$K'_a(x) = U(\Lambda) K_a(x) U^{-1}(\Lambda) = \Lambda_a{}^b K_b(x). \tag{D13}$$

So the explicitly $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ version of h goes as

$$h' = \begin{pmatrix} s_\ell^a \Lambda_a{}^b K_b & 0 \\ 0 & s_r^a \Lambda_a{}^b K_b \end{pmatrix} \;=\; D^{-1\dagger} h D^{-1} = \begin{pmatrix} D_\ell^{-1\dagger} s_\ell^b D_\ell^{-1} K_b & 0 \\ 0 & D_r^{-1\dagger} s_r^b D_r^{-1} K_b \end{pmatrix} \tag{D14}$$

since

$$\begin{aligned} D_\ell^{-1\dagger} s_\ell^b D_\ell^{-1} K_b &= s_\ell^a \Lambda^{-1b}{}_a K_b = s_\ell^a \Lambda_a{}^b K_b \\ D_r^{-1\dagger} s_r^b D_r^{-1} K_b &= s_r^a \Lambda^{-1b}{}_a K_b = s_r^a \Lambda_a{}^b K_b. \end{aligned} \tag{D15}$$

Appendix E. Hermitian form of the Dirac action

The fully expanded covariant derivative $D_i\psi$ is

$$D_i\psi = \left(\partial_i - \mathrm{i}\,\frac{1}{2}\,\boldsymbol{r}_i \cdot \boldsymbol{\sigma}\, I - \frac{1}{2}\,\boldsymbol{b}_i \cdot \boldsymbol{\sigma}\gamma^5\right)\psi. \tag{E1}$$

If $\bar{\psi} = \psi^\dagger\beta$, then $-\bar{\psi}\;\gamma^a c_a{}^i D_i\psi$ is

$$\begin{aligned}-\psi^\dagger\beta\;\gamma^a c_a{}^i D_i\psi &= -\psi^\dagger\beta\;\gamma^a\;c_a{}^i(\partial_i\psi) - \psi^\dagger\beta\;\gamma^a\;c_a{}^i(-i\frac{1}{2}\,\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I)\psi - \psi^\dagger\beta\;\gamma^a\;c_a{}^i(-\frac{1}{2}\,\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\gamma^5)\psi\\ &= -\psi^\dagger\beta\;\gamma^a\;c_a{}^i(\partial_i\psi) + \frac{1}{2}\;i\;\psi^\dagger\beta\gamma^a\;c_a{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I\psi + \frac{1}{2}\;\psi^\dagger\beta\;\gamma^a\;c_a{}^i\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\gamma^5\psi.\end{aligned} \tag{E2}$$

Its adjoint $[-\bar{\psi}c_a{}^i\gamma^a D_i\psi]^\dagger$ is

$$\begin{aligned}[-\psi^\dagger\beta\gamma^a c_a{}^i D_i\psi]^\dagger &= [-\psi^\dagger\beta\gamma^a c_a{}^i(\partial_i\psi)]^\dagger + \left[\frac{1}{2}i\psi^\dagger\beta\gamma^a c_a{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}I\psi\right]^\dagger + \left[\frac{1}{2}\psi^\dagger\beta\gamma^a c_a{}^i\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\gamma^5\psi\right]^\dagger\\ &\equiv A_1 + A_2 + A_3.\end{aligned} \tag{E3}$$

Now $\beta\,\gamma^a = \mathrm{i}\gamma^0\gamma^a$ is antihermitian, so the first term is

$$A_1 = [-\psi^\dagger\beta\gamma^a c_a{}^i(\partial_i\psi)]^\dagger\; = (\partial_i\psi^\dagger)\beta\;\gamma^a\;c_a{}^i\;\psi = -\psi^\dagger\;\beta\;\gamma^a(\partial_i c_a{}^i)\;\psi - \psi^\dagger\;\beta\;\gamma^a\;c_a{}^i\;\partial_i\psi. \tag{E4}$$

The second term is

$$A_2 = \left[\frac{1}{2}i\psi^\dagger\beta\gamma^a c_a{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}I\psi\right]^\dagger = \frac{1}{2}\;i\;\psi^\dagger\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I\beta\;\gamma^a c_a{}^i\;\psi = \frac{1}{2}\;i\;\psi^\dagger\;\beta\;\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I\gamma^a c_a{}^i\;\psi. \tag{E5}$$

The third term is

$$A_3 = \left[\tfrac{1}{2}\psi^\dagger\beta\gamma^a c_a{}^i\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\gamma^5\psi\right]^\dagger = -\tfrac{1}{2}\psi^\dagger\;\gamma^5\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\beta\;\gamma^a\;c_a{}^i\;\psi = -\tfrac{1}{2}\psi^\dagger\beta\;\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\gamma^a\;c_a{}^i\;\gamma^5\;\psi.$$

Since $[\sigma^s,\,\gamma^j] = 2i\,\epsilon_{sjk}\gamma^k$, we have

$$\sigma^s\;\gamma^j = \gamma^j\;\sigma^s + 2i\;\epsilon_{sjk}\gamma^k. \tag{E6}$$

So the $\boldsymbol{r}$ terms are

$$\begin{aligned}\frac{1}{4}\;i\;\psi^\dagger\beta\gamma^a\;c_a{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I\psi + \frac{1}{4}\;i\;\psi^\dagger\;\beta\;\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I\gamma^a c_a{}^i\;\psi &= \frac{1}{4}\;\psi^\dagger(2c_0{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I + \beta\;c_j{}^i r_i{}^s\{i\gamma^j,\,\sigma^s\})\psi\\ &= \frac{1}{2}\psi^\dagger(c_0{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I + \beta\;c_s{}^i r_i{}^s\gamma^5\beta)\psi = \frac{1}{2}\psi^\dagger(c_0{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I - c_s{}^i\;r_i{}^s\gamma^5)\psi,\end{aligned} \tag{E7}$$

while the $\boldsymbol{b}$ terms are

$$\frac{1}{2}\;\psi^\dagger\beta\gamma^a\;c_a{}^i\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\gamma^5\psi - \frac{1}{2}\psi^\dagger\beta\;\boldsymbol{b}_i\cdot\boldsymbol{\sigma}\gamma^a\;c_a{}^i\;\gamma^5\;\psi = i\;\epsilon_{tsk}\;c_t{}^i\;b_i{}^s\;\psi^\dagger\;\beta\;\gamma^k\;\gamma^5\;\psi. \tag{E8}$$

The hermitian Dirac action density then is

$$= -\psi^\dagger\beta c_a{}^i\gamma^a\partial_i\psi - \frac{1}{2}\psi^\dagger\;\beta\;\gamma^a(\partial_i c_a{}^i)\;\psi + \psi^\dagger\frac{1}{2}(c_0{}^i\boldsymbol{r}_i\cdot\boldsymbol{\sigma}\;I - c_s{}^i r_i{}^s\gamma^5)\psi - \frac{1}{2}\epsilon_{jsk}\;c_j{}^i b_i{}^s\psi^\dagger\sigma^k\psi. \tag{E9}$$

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