Generalized Optical Theorems and Steinmann Relations*

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Generalized optical theorems and generalized Steinmann relations for $3 \to 3$ processes are described. These results follow from the field-theoretic formalism of Bros, Epstein, and Glaser. The theorems express in terms of physical-region scattering functions the discontinuity of any $3 \to 3$ scattering function across any basic cut, which is a channel-energy cut that runs from the lowest normal threshold to plus infinity. Previously these discontinuities had been expressed, in general, only in terms of the analytic continuations of scattering functions. A remarkable property is found: The discontinuities have the algebraic structure they would have if the basic cuts were cuts in independent variables. Moreover, a generalized version of the Steinmann relations holds: The double discontinuity across any pair of *independently treated* crossed basic cuts vanishes. The original Steinmann relations have a much narrower scope. The generalized Steinmann relations are used to express compactly in terms of physical-region scattering functions any single, double, or higher-order multiple discontinuity of any $3 \to 3$ scattering function across any combination of basic cuts, evaluated on any specified side of each of the remaining basic cuts. All these basic discontinuities are expected to enjoy Regge asymptotic behavior. Although only basic cuts are mentioned, there is no neglect of the effects of other cuts.

GENERALIZED OPTICAL THEOREMS AND STEINMANN RELATIONS

Recent developments in the theory of high-energy reactions have been exploiting to an increasing extent the physical-region discontinuity structure of multiparticle scattering amplitudes. This structure emerges from the basic assumptions of either field theory or S-matrix theory, and imposes precise and rigorous constraints on the dynamics of high-energy processes.

This line of development was initiated by Mueller,[1] who showed in effect how important properties of high-energy inclusive cross sections can be derived by combining the inclusive optical theorem with the assumption that the basic discontinuities of multiparticle scattering functions enjoy Regge asymptotic behavior. The inclusive optical theorem, which is one element of the physical-region discontinuity structure, equates the inclusive cross section for the process $a+b \to c+$anything to a multiple of a certain basic discontinuity of the scattering function for the process $a+b+c \to a+b+c$. The basic discontinuities are discontinuities across basic cuts, which are the normal-threshold cuts that start at the lowest normal threshold in the channel energy and extend to plus infinity.

The extensive developments initiated by Mueller's work have so far exploited mainly the particular discontinuity formula embodied in the inclusive optical theorem. However, the assumed Regge behavior places theoretical constraints on all basic discontinuities. These further constraints have already been partially exploited. For example, DeTar and Weis[2] have used them, together with the Steinmann relations, to deduce the form of double- and triple-Regge vertices, and Weis[3] has noted important physical consequences of the associated Regge factorization property. However, to fully exploit the Regge properties of all the basic discontinuities it is probably necessary to have expressions for them in terms of physical scattering functions. For it is these expressions that directly link the theoretical constraints to the basic physical and theoretical quantities of the theory.

A general formula for basic discontinuities has been derived by us in an earlier work,[4] and this formula has proved useful.[5] However, it has three undesirable features:

(1) It expresses the basic discontinuities in terms of analytic continuations of scattering functions, rather than in terms of the physical scattering functions themselves.

(2) It covers, for the $3 \to 3$ case, only 2282 of the 65 536 possible functions.

(3) It is an off-mass-shell result.

These three points will now be discussed in more detail.

The first point is that the basic discontinuity equation obtained earlier generally expresses discontinuities in terms of various unphysical boundary values of analytic continuations of scattering functions, instead of in terms of the physical scattering amplitudes themselves. Thus the formula involves a host of new unphysical functions, namely, the unphysical boundary values of the various scattering functions in terms of which the discontinuities are expressed.

To avoid introducing these unphysical functions, we have used the formalism of Bros, Epstein, and Glaser[6,7] to express all basic discontinuities of all $3\to 3$ processes in terms of physical scattering amplitudes alone. The derivation of these formulas, which are generalizations of the ordinary and inclusive optical theorems, will be presented in a later work.

The second point is that the basic discontinuity formula derived earlier covers, for the $3\to 3$ case, only 2282 of the 65 536 possible functions, as will now be explained. The physical region of an arbitrary $3\to 3$ amplitude is cut by 16 basic cuts: one total-energy cut, three initial subenergy cuts, three final subenergy cuts, and nine cross-energy cuts. If one could *independently* specify whether the limit was to be taken from above or below each of these 16 cuts, then there would be $2^{16}=65\,536$ possibilities. Stated differently, if these 16 cuts lay in a space of 16 corresponding dimensions, then there would be 65 536 functions, one corresponding to each possible combination of sides of these cuts.

The 16 basic cuts are not independent in this sense, for the 16 channel energies are linear combinations of the energies of the six particles of the problem, and these six energies are themselves subject to the requirement of energy conservation. Consequently, there are not 65 536 regions, but only 2282: The 16 channel-energy cuts divide the five-dimensional space of complex energies into 2282 disjointed regions. These regions are called *cells*. The Bros-Epstein-Glaser formalism, upon which the earlier work was based, deals specifically with those real boundary values that can be defined as limits taken from points within these cells. Accordingly, the basic discontinuity equation derived earlier covers only those discontinuities that are defined as the differences of the real boundary values taken from adjacent cells.

This limitation to discontinuities formed from differences of the 2282 cell functions is awkward. For it means that before applying the basic discontinuity formula, one must check to see whether there is a pair of adjacent cells that corresponds to the discontinuity in question. It turns out, for example, that there is no cell that lies below just one single subenergy cut and above all other cuts.

The number of basic discontinuities expressed by the earlier formula, i.e., the number of pairs of adjacent cells, is far smaller than the number 16×2^{15} that would arise from evaluating the discontinuities across each of 16 *independent* cuts on each side of each of the other 15 remaining independent cuts. However, it is a remarkable fact that the basic discontinuity equations for the $3\to 3$ case, when expressed in terms of physical scattering functions, extend coherently to all 16×2^{15} discontinuities, i.e., if all the unphysical boundary values occurring in the $3\to 3$ discontinuity formulas are expressed in terms of physical scattering functions, then the basic discontinuity equation, as stated in Ref. 4, can be applied to all 16×2^{15} discontinuities *without* regard to the cell-function limitation. The many expressions that can be obtained from these discontinuity formulas for each of the 65 536 functions all agree by virtue of the unitarity relations. This means that the 16 basic cuts can be treated, insofar as these discontinuities are concerned, just as if they were cuts in 16 independent variables.

This result implies the validity of generalized Steinmann relations. The original Steinmann relations, which assert that the double discontinuity across any pair of crossed cuts vanishes, were derived[8] in a framework that strictly enforces the constraints that link the various channel energies. Thus the original Steinmann relations cover only those double discontinuities that are formed from differences of cell functions. For the $3\to 3$ case, in particular, the original Steinmann relations cover only those double discontinuities in which each of the four functions used to form the double discontinuity is one of the 2282 cell functions. These four cell functions must correspond to four cells that all lie on the same sides of all the other basic cuts.

In recent work on high-energy scattering processes important use has been made of a generalized version of the Steinmann relations. In these generalized Steinmann relations the basic cuts are treated as if they were cuts in independent variables, and the vanishing of the double discontinuity across pairs of crossed cuts is imposed without regard to the cell-function limitation. DeTar and Weis[2] have used these generalized Steinmann relations to deduce properties of double- and triple-Regge vertices. Halliday[9] has used them to check the compatibility of multi-Regge amplitudes with unitarity. Weis[10] has used them to show factorization properties of multi-Regge amplitudes. Cardy and White[11] have used them to show that the Pomeron–particle-particle vertex that controls total cross sections need not vanish.

The generalized Steinmann relations can be regarded as an abstraction from dual-resonance models, in which they hold. Or they can be regarded as applications of a conjecture by Olive.[12] However, the rigorous status of dual-resonance models is not yet clarified, and Olive's conjecture was formulated in a framework in which unitarity was considered only below the four-particle threshold, and all non-normal threshold cuts were ignored. These approximations are not justified at high energies. Our results show that the generalized Steinmann relations for $3\to 3$ processes follow without approximation directly from basic field-theoretic principles.

The generalized Steinmann relations allow the discontinuity formulas for $3\to 3$ processes to be summarized in a compact way. Let J represent any subset of the set of 16 indices j that label the 16 basic cuts. By virtue of the independent-cut structure, one can define M^J to be the scattering function evaluated below each of the cuts $j\in J$ and above each of the remaining cuts. Here above and below are defined with reference to the *physical* side: The physical scattering function is defined to lie *above* all cuts. Thus $M\equiv M^{\phi}$, with ϕ the empty set, is the physical scattering function.

Let $Jk\equiv J\cup\{k\}$. Then the function $M^J{}_k$ defined by

$$M^J{}_k\equiv M^J-M^{Jk} \tag{1.1}$$

is the discontinuity across the cut labeled by k, evaluated below all the cuts labeled by the $j\in J$ and above all the remaining cuts. The terminology used here is that appropriate to the situation in which all 16 cuts are independent.

Let K be a set of indices j, and define

$$M^J{}_{Kk}\equiv M^J{}_K-M^{Jk}_K\,. \tag{1.2}$$

Then, by induction on K, the function $M^J{}_{Kk}$ is the multiple discontinuity across the set of cuts labeled by the set $Kk\equiv K\cup\{k\}$, evaluated below all the cuts labeled by the $j\in J$ and above all the remaining cuts. As usual, the multiple discontinuity across a set of n cuts is defined to be the sum of the 2^n boundary values corresponding to the regions lying on all possible sides of these cuts, each with a factor $(-1)^m$, where m is the number of cuts below which the corresponding region lies. Thus a double discontinuity is $M(+,+)-M(+,-)-M(-,+)+M(-,-)$, where the sign arguments indicate the sides of the two cuts.

By writing Eq. (1.2) in an inverted form, one may inductively derive the formula

$$M^J{}_K=\sum_{J_\alpha\subset J}(-1)^{n_\alpha}M_{KJ_\alpha}\,. \tag{1.3}$$

Here the sum is over all (empty or nonempty) subsets J_α of J, n_α is the number of indices in the set J_α, and $KJ_\alpha\equiv K\cup J_\alpha$. This relation expresses all 65 536 functions M^J, all 16×2^{15} single discontinuities $M^J{}_K$, and all of the numerous multiple discontinuities $M^J{}_K$ in terms of the physical scattering function M, the 16 single discontinuities M_j, and the various multiple discontinuities M_{jk}, M_{jkl}, etc. For the $3\to 3$ case, all multiple discontinuities $M_{j\ldots l}$ with more than three indices vanish, by virtue of the generalized Steinmann relations, as do most double and triple discontinuities. Thus Eq. (1.3) expresses the entire set of functions $M^J{}_K$ in terms of the relatively few nonvanishing functions M, M_j, M_{jk}, and M_{jkl}.

Explicit formulas for this small group of nonvanishing functions will be given later. First, however, the third undesirable feature of the normal discontinuity formula of Ref. 4 is discussed.

This third feature is that the results are essentially off-mass-shell results. In particular, the primitive domain of analyticity does not intersect the mass shell. The real physical points lie on the boundary of this domain of analyticity, and must be approached through off-mass-shell regions of analyticity. In fact, all of the various physical and unphysical real boundary values are expressed as limits of analytic functions from off-mass-shell domains of analyticity.

Bros, Epstein, and Glaser[13] have made important progress in extending the primitive domains of analyticity into the mass shell. However, we do not follow that line here, but merely note that mass-shell analogs of all the results described here were derived from purely S-matrix principles prior to the inception of the present work. In fact, the S-matrix results go considerably beyond the results described here. For in the present work no analyticity properties are obtained for any of the 65 536 functions M^J that are not among the original 2282. In the S-matrix approach each of the 65 536 functions M^J is constructed to have analyticity properties appropriate to a function that lies below all the basic cuts labeled by $j\in J$ and above all remaining basic cuts. These mass-shell analyticity properties of the M^J will be described and proved elsewhere.

As discussed above, all the functions $M^J{}_K$ in the $3\to 3$ case are compactly expressed by Eq. (1.3) in terms of the functions M, J_j, M_{jk}, and M_{jkl}. These latter functions are now displayed explicitly. A diagrammatic notation is used in which the basic definitions are given in Eqs. (1.4) shown in Fig. 1.

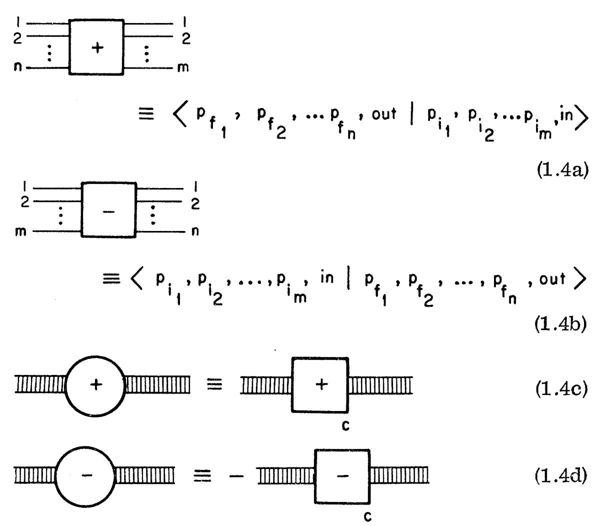


FIG. 1. Diagrammatic equations (1.4a)–(1.4d).

The subscript c means *connected part*. An external shaded strip represents an arbitrary set of lines. An internal shaded strip indicates a sum over a complete set of *in* or *out* states, with the same weighting as in the expansion of the identity operator I:

$$I=\sum_n |n,\pm\rangle\langle n,\pm|, \tag{1.5}$$

where + and − designate out and in, respectively.

In terms of this notation, the function M is represented by a plus bubble as in Eq. (1.4c). All 16 single discontinuities M_j are given by the single formula in Eq. (1.6) (see Fig. 2), where the division of external lines between the two plus bubbles is the division defined by j. In particular, if the 16 sets j are labeled by

$$\bar{t}\equiv\{4,5,6\},$$

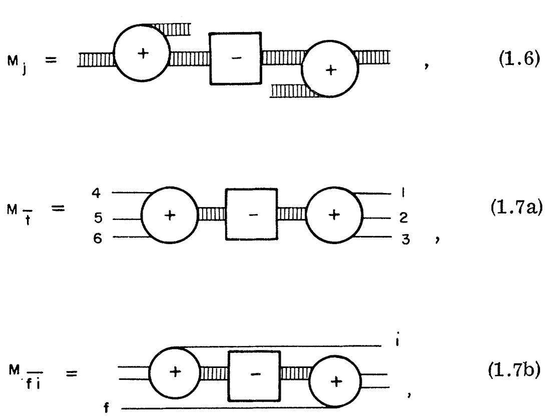
$$\bar{f}\equiv\bar{t}-\{f\},$$

$$\bar{i}\equiv\bar{t}+\{i\},$$

and

$$\overline{fi}\equiv\bar{t}+\{i\}-\{f\},$$

then the M_j are as given in Eqs. (1.7). The last two formulas can be simplified by means of the identities in Eqs. (1.8a) and (1.8b), which follow directly from the unitarity relations in Eqs. (1.9a) and (1.9b). Here the I box represents the identity operator I and satisfies for any operator X the identities in Eqs. (1.10a) and (1.10b). The I box also satisfies the rules in Eqs. (1.11a) and (1.11b), where Eq. (1.11c) is used for the inner product of two single-particle states. These identities will be used later.



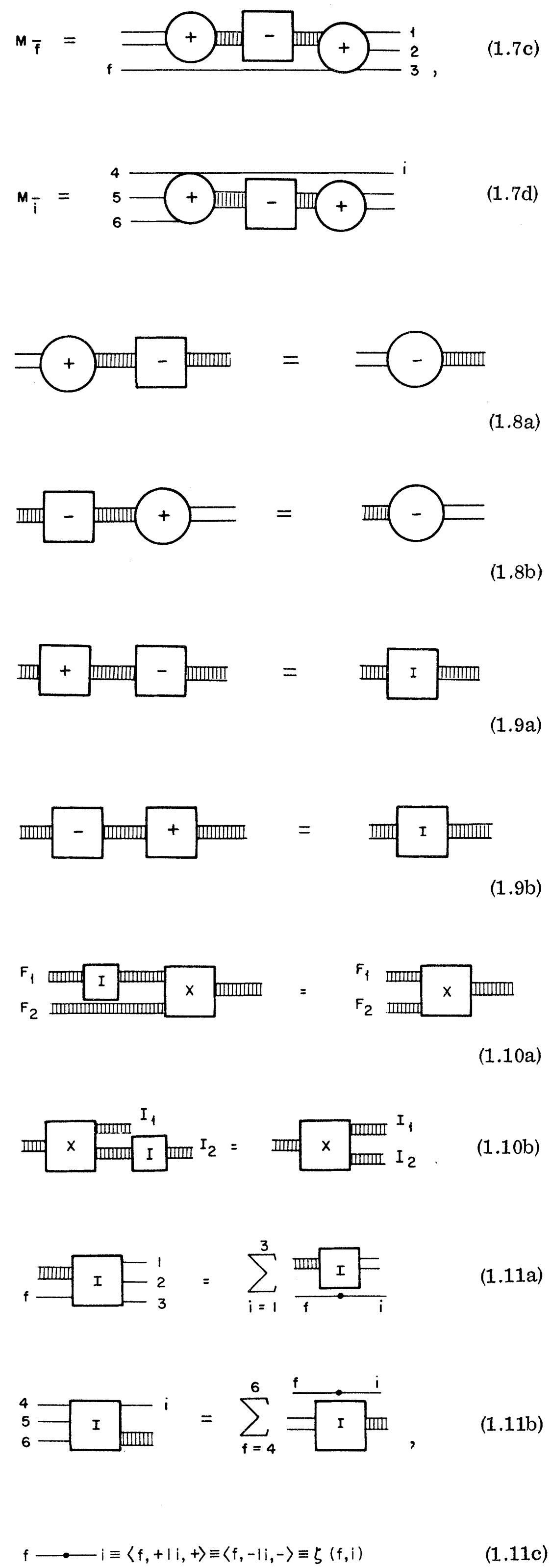
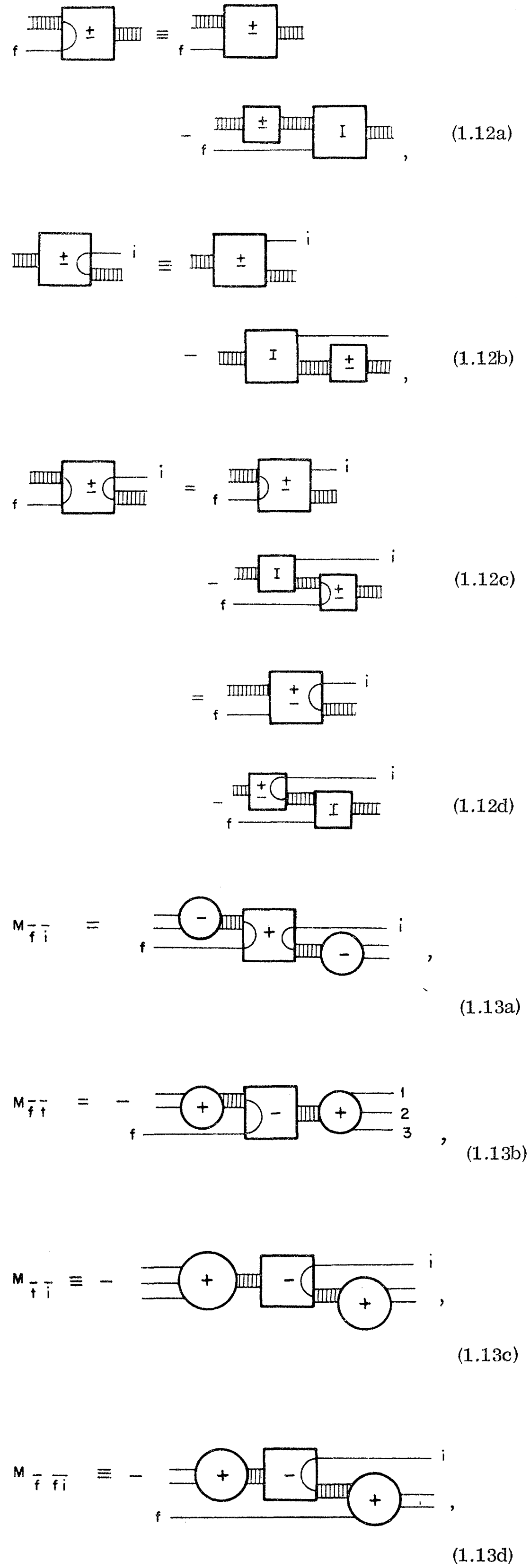


FIG. 2. Diagrammatic equations (1.6)–(1.11).



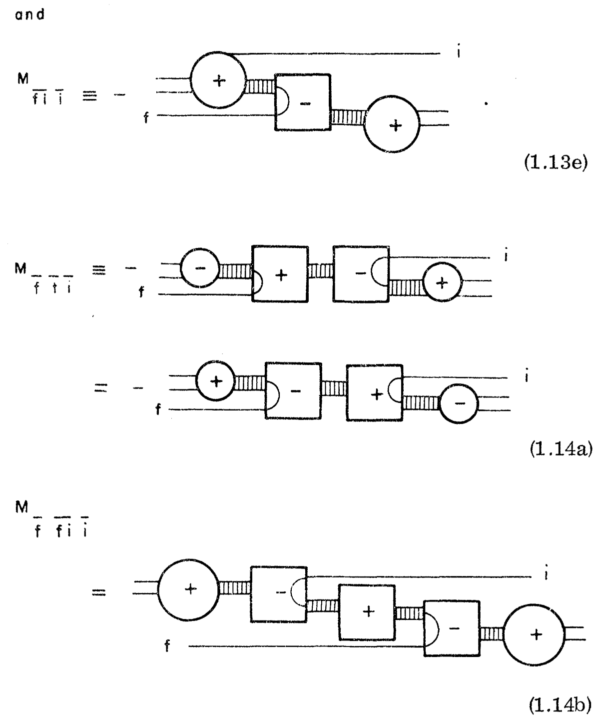


FIG. 3. Diagrammatic equations (1.12)–(1.14).

The double and triple discontinuities are defined in terms of the quantities defined in Eqs. (1.12a)–(1.12d) (see Fig. 3). Equivalently, these quantities can be defined as those terms in the cluster expansion of the plus or minus boxes in which the distinguished line f or i is connected to some other final or initial line, respectively.

The nonzero double discontinuities M_{jk} are given in Eqs. (1.13a)–(1.13e). The nonzero triple discontinuities M_{ijk} are given in Eqs. (1.14a) and (1.14b). These explicit equations, when combined with Eq. (1.3), give a complete representation of the basic-cut structure of the $3 \to 3$ scattering amplitude.

By considering the Hermitian conjugate of Eq. (1.3), one can obtain alternative forms of the discontinuity formulas (1.7), (1.13), and (1.14). Since Hermitian conjugation reverses the sign inside each box or bubble and multiplies each bubble by minus one, the signs inside all boxes and bubbles will be reversed and *above* will be interchanged with *below*. Taking into account the extra minus sign in the definition (1.4d) of the minus bubble, one obtains the general rule

$$(M^J{}_K)^\dagger = (-1)^{n+1} M^{E-J}_{K}{}^{-K}, \tag{1.15}$$

where n is the number of elements of K, and E is the complete set of 16 indices.

The 67 discontinuities given by Eqs. (1.7), (1.13), and (1.14) can all be expressed in terms of the 16 single discontinuities (1.7), the nine inclusive-cross-section discontinuities in Eq. (1.16) (see Fig. 4), and the nine double discontinuities (1.13a), together with the Hermitian conjugates of these 34 discontinuities. Direct application of the defining properties (1.10), (1.11), and (1.12), plus unitarity, give the results shown in Eqs. (1.17a)–(1.17d). The equations for $M_{\overline{fi}\,\bar{i}}$ and $M_{\bar{i}\,\bar{i}}$ are reflections through the origin of (1.17c) and (1.17d), with i and f interchanged.

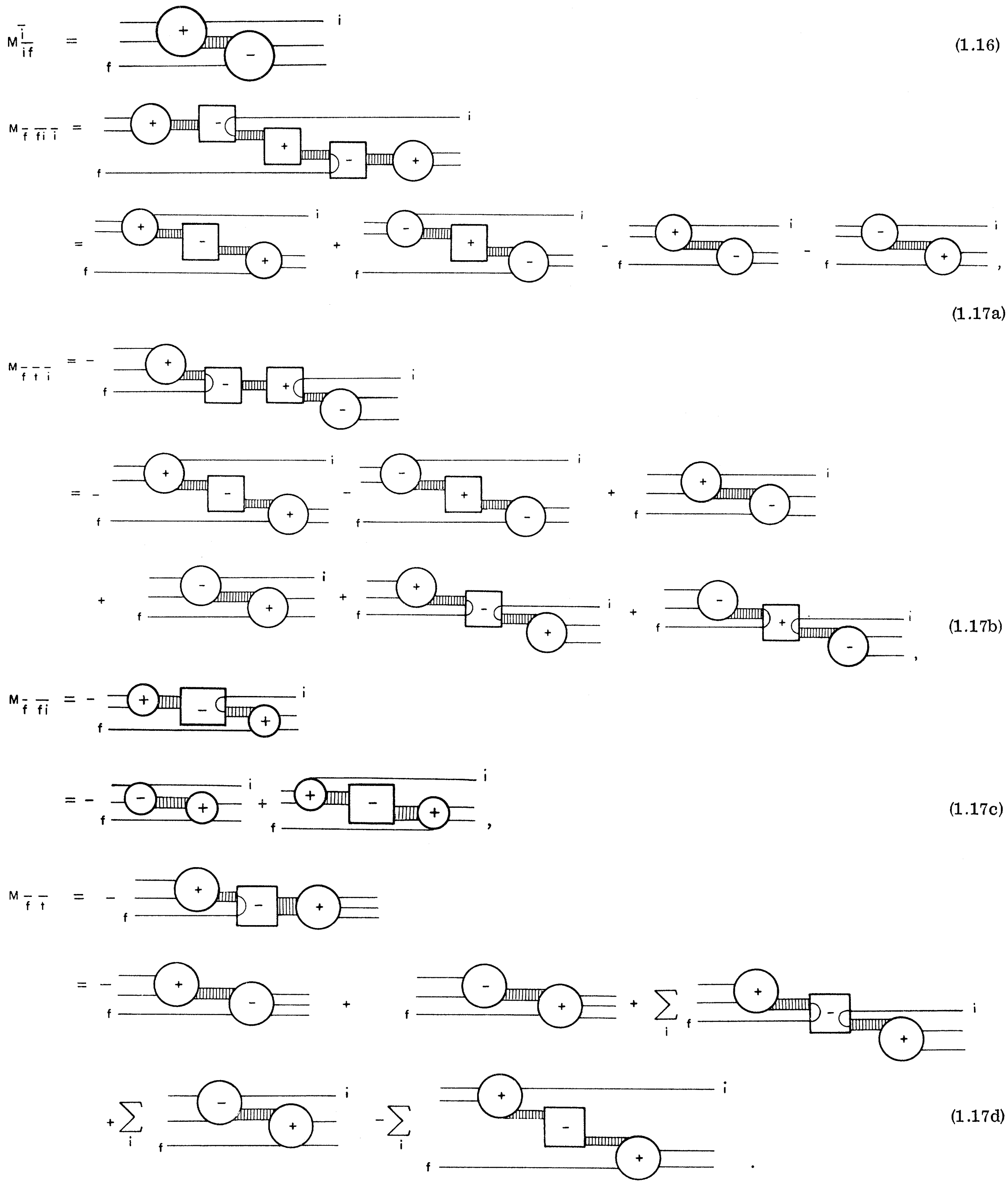


FIG. 4. Diagrammatic equations (1.16) and (1.17).

The formulas (1.17) for the various double and triple discontinuities can also be derived directly from the independent-cut structure and the generalized Steinmann relations, together with Eq. (1.15) and the discontinuity formulas (1.6), (1.16), and (1.13a). Equations (1.6) and (1.16) are expressions for basic discontinuities that have been derived previously.[15] Thus the principal new results of this work are contained in the double discontinuity formula (1.13a), together with the result that the independent-cut structure and generalized Steinmann relations follow for the $3 \rightarrow 3$ case from basic field-theoretic principles.

By virtue of the above results, the apparent complexity of the $3 \rightarrow 3$ discontinuity structure, suggested by the large number 65 536 of different functions M^J, is greatly reduced. All the M^J are built from M and 68 elementary discontinuities, which are themselves special cases of the three simple formulas (1.6), (1.16), and (1.13a) and their complex conjugates.

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