An effective potential that is real

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In theories with spontaneous symmetry breaking, the exact effective potential $V(\phi_c, T)$ is real, but its loop expansion can be complex. A generalization of the effective potential is developed that is real and that can be computed perturbatively. For the theory with the classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$, this real effective potential closely tracks the usual effective potential where the latter is real, $|\phi_c| \geq \sigma/\sqrt{3}$, and at finite temperatures displays at $\phi_c = 0$ a local minimum, which may have astrophysical implications. The critical temperature at the one-loop level runs from $T_C \approx 1.81\sigma$ for $\lambda = 0.1$ to $T_C \approx 1.74\sigma$ for $\lambda = 1$.

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I. INTRODUCTION

The effective potential was introduced by Heisenberg and Euler [1] and by Schwinger [2]. Goldstone, Salam, and Weinberg [3] and Jona-Lasinio [4] developed the effective potential and applied it to the problem of symmetry breaking [5]. Coleman and Weinberg used it to show that radiative corrections could break symmetries [6]. Linde [7] and Weinberg [8] later used it to obtain a lower bound on the mass of the Higgs boson. West and others have used it to study the breaking of supersymmetry [9].

The possibility that broken symmetries might be restored at high temperatures was raised by Kirzhnits and Linde [10] and confirmed by them [11], by Dolan and Jackiw [12], and by Weinberg [13], who with Bernard [14] introduced and developed the finite-temperature effective potential. Much current work on the early universe is based upon the finite-temperature effective potential [15].

Although the effective potential has had a long history of successful applications to particle physics, it does not seem to be well suited to theories that exhibit spontaneous symmetry breaking. In such theories the exact effective potential is real [16], but its perturbative series can be complex [17]. In the example provided by the symmetry-breaking classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$, the loop expansion of the finite-temperature effective potential is complex at all temperatures T for $|\phi_c| < \sigma/\sqrt{3}$. In such theories the accuracy of the one-loop effective potential does not extend down to the small values of $|\phi_c|$ that are of interest in studies of the early universe. And where the effective potential is complex, it is ambiguous as an approximation to a free-energy density — although it may be interpreted as a decay rate [18].

Because of this limitation of the perturbative effective potential, some physicists have turned to nonperturbative techniques. Chang [19] has invented a variational method called the gaussian effective potential, which Barnes and Ghandour [20] and Stevenson [21] have developed. Fukuda and Kyriakopoulos [22] have introduced a version of the effective potential that is well suited to lattice computations; O'Raifeartaigh, Wipf, and Yoneyama [23] have analyzed this potential. Ringwald and Wetterich [24] have suggested the use of block-spin techniques.

The goal of the present paper is to generalize the effective potential so that it can be applied simply and perturbatively to theories with spontaneous symmetry breaking. The usual effective potential is the Legendre transform of the Helmholtz free-energy density for the modified Hamiltonian $H + \int j\phi \, d^3x$, in which j is an external source. For theories in which $V''(\phi)$ takes on negative values, as it must when $V(\phi)$ has two minima, the linear probe $j\phi$ should be generalized to a quadratic polynomial $jP(\phi)$. This advice has been given in the past with varying degrees of obliqueness by Cornwall, Jackiw, and Tomboulis [25], by Hawking and Moss [26], and by Lawrie [27], but it has not been followed. When discussing theories with spontaneous symmetry breaking, most physicists either ignore the complexity of the usual effective potential [15] or work in a region of parameter space in which scalar loops can be ignored [28].

In what follows I shall discuss the case of a single scalar field ϕ interacting with an arbitrary renormalizable potential $V(\phi)$. If the curvature $V''(\phi)$ of the potential is positive, then the usual effective potential with a linear probe $j\phi$ is optimal. But if the curvature $V''(\phi)$ of the potential is negative for some values of the field ϕ, then a quadratic polynomial $jP(\phi)$ should be used. If the potential $V(\phi)$ of indefinite curvature has a single minimum at ϕ_1, then a suitable probe is $P(\phi) = (\phi - \phi_1)^2/2$. If the potential $V(\phi)$ of indefinite curvature has two minima separated by a local maximum at ϕ_0, then $P(\phi) = (\phi - \phi_0)^2/2$ is an appropriate probe. Such quadratic probes $jP(\phi)$ lead to effective potentials that possess real loop expansions.

For the classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$, which has a local maximum at $\phi_0 = 0$, the right probe is $P(\phi) = \phi^2/2$. The resulting real effective potential

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closely tracks the usual effective potential where the latter is real and naturally extends down to $\phi_c = 0$, where it has a local minimum. The reflection symmetry of the action is restored to the vacuum at a critical temperature T_C, which runs from $T_C \approx 1.81\sigma$ for $\lambda = 0.1$ to $T_C \approx 1.74\sigma$ for $\lambda = 1$, a range of values that may be compared with Weinberg's [13] result $T_C \approx 2\sigma$. The first few terms of the high-temperature expansion of the one-loop, real effective potential $V_1(\phi_c, T; P)$ with probe $P(\phi) = \phi^2/2$ are

$$V_1(\phi_c, T; P) = -\frac{\pi^2}{90}T^4 + \frac{\lambda}{24}\left(3\phi_c^2 - \sigma^2\right)T^2 + \frac{\lambda^{3/2}}{12\sqrt{2}\,\pi}\left(3\sigma^2|\phi_c| - 7|\phi_c|^3\right)T. \tag{1}$$

The corresponding terms of the usual effective potential are

$$V_1(\phi_c, T) = -\frac{\pi^2}{90}T^4 + \frac{\lambda}{24}\left(3\phi_c^2 - \sigma^2\right)T^2 - \frac{\lambda^{3/2}}{12\pi}\left(3\phi_c^2 - \sigma^2\right)^{3/2}T. \tag{2}$$

These two expansions possess the same two leading terms, but they differ in the third term, which is imaginary for $|\phi_c| < \sigma/\sqrt{3}$ in the case of the usual effective potential. In the expansion of the real effective potential $V_1(\phi_c, T; P)$, the term $\lambda^{3/2}\sigma^2|\phi_c|T$ occurs with a positive coefficient and creates a local minimum at $\phi_c = 0$ for $T > 0$. This dip may have astrophysical implications [29]: it may trap the classical field at $\phi_c = 0$ long enough for cosmic inflation to ensue.

The traditional effective potential is discussed in Secs. II–V in a pedagogical manner inspired by Weinberg [13]. The real effective potential is introduced in Sec. VI. The meaning of effective potentials is discussed in Sec. VII. In Sec. VIII the computation of the real effective potential is discussed for the case of an arbitrary renormalizable classical potential $V(\phi)$. This computation is carried out in detail for the potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$ in Sec. IX.

II. THE PARTITION FUNCTION OF A FREE FIELD

One of the clearest descriptions of the finite-temperature effective potential is Weinberg's operator formulation [13]. Stripped of fermions and gauge fields and reduced to a single scalar field, it will serve as the basis for the introductory sections of this paper.

The basic quantity of statistical physics is the partition function Z of the inverse temperature $\beta = 1/T$, which for a system described by the Hamiltonian H is the trace

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}. \tag{3}$$

All the quantities of this paper will be expressed in terms of the partition function $Z(\beta)$ for a free, real scalar field ϕ of mass m with the hamiltonian

$$H = \int \tfrac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] d^3x = \sum_k \omega_k \left[a^\dagger(k)a(k) + \tfrac{1}{2}\right] , \tag{4}$$

where $\omega_k = \sqrt{k^2 + m^2}$. By inserting a complete set of energy eigenstates, one may find for the partition function $Z(\beta)$ the expression

$$Z(\beta) = \prod_k e^{-\frac{\beta\omega_k}{2}} \sum_{n_k=0}^{\infty} e^{-\beta n_k \omega_k} = \prod_k \frac{e^{-\frac{\beta\omega_k}{2}}}{\left(1 - e^{-\beta\omega_k}\right)} \tag{5}$$

which is simpler as a logarithm:

$$-\ln Z(\beta) = \sum_k \left(\frac{\beta\omega_k}{2} + \ln\left(1 - e^{-\beta\omega_k}\right)\right) = \int \frac{L^3 d^3k}{(2\pi)^3}\left[\frac{\beta\omega_k}{2} + \ln\left(1 - e^{-\beta\omega_k}\right)\right] , \tag{6}$$

where L^3 is the volume of quantization.

III. THE EFFECTIVE POTENTIAL

For a scalar field ϕ described by a hamiltonian H perturbed by an external current j, the Helmholtz free-energy density $A(j, T)$ is defined as

$$\exp[-\beta L^3 A(j, T)] = \mathrm{Tr}\, \exp\left[-\beta\left(H + j\int \phi(x)\, d^3x\right)\right] . \tag{7}$$

The free energy $A(j, T)$ is therefore proportional to the logarithm of the partition function $Z(\beta, j)$,

$$A(j, T) = -\frac{\ln Z(\beta, j)}{\beta L^3} , \tag{8}$$

for the system described by the perturbed hamiltonian $H + j\int \phi(x) d^3x$.

The mean value of the field ϕ is a function of the current j,

$$\phi_c(j, T) \equiv \langle\phi\rangle_j = \frac{\mathrm{Tr}\,\phi(x)\exp\left[-\beta\left(H + j\int\phi(x)d^3x\right)\right]}{\mathrm{Tr}\,\exp\left[-\beta\left(H + j\int\phi(x)d^3x\right)\right]} , \tag{9}$$

and is a derivative of the Helmholtz potential

$$\phi_c(j, T) = \frac{\partial A(j, T)}{\partial j}. \tag{10}$$

The finite-temperature effective potential $V(\phi_c, T)$ is defined [10–12] as a Legendre transform of the Helmholtz potential

$$V(\phi_c, T) \equiv A(j, T) - j\,\frac{\partial A(j, T)}{\partial j} = A(j, T) - j\phi_c \tag{11}$$

expressed as a function of the "classical field" ϕ_c:

$$V(\phi_c, T) = A(j(\phi_c, T), T) - j(\phi_c, T)\phi_c \tag{12}$$

rather than of the current j. The effective potential may be thought of as a Gibbs free-energy density. It is obviously real.

The utility of the effective potential derives from its ability at its minima to represent the unperturbed system. For from Eqs. (10) and (11), it follows that the derivative of the effective potential with respect to the classical field ϕ_c is proportional to the external current j:

$$\frac{\partial V(\phi_c,T)}{\partial \phi_c} = \frac{\partial j}{\partial \phi_c}\left(\frac{\partial A(j,T)}{\partial j} - \phi_c\right) - j = -j. \tag{13}$$

Thus the current j must vanish at the stationary points of $V(\phi_c,T)$:

$$0 = \frac{\partial V(\phi_c,T)}{\partial \phi_c} = -j. \tag{14}$$

At zero temperature, the minimum value of the effective potential is the energy density of the ground state of the system.

Since it is through the factor $\exp[-\beta j \int \phi(x) d^3x]$ that the current j influences the mean value ϕ_c, the relationship between the current j and the mean value ϕ_c is inverse. By differentiating Eq. (9) with respect to the source j, one finds explicitly

$$\frac{\partial \phi_c(j,T)}{\partial j} = -\beta \langle [\phi - \phi_c(j,T)]^2 \rangle_j < 0. \tag{15}$$

So by differentiating the formula (13) with respect to ϕ_c, one sees that the effective potential has a non-negative second derivative:

$$\frac{\partial^2 V(\phi_c,T)}{\partial \phi_c^2} = -\frac{\partial j}{\partial \phi_c} \geq 0. \tag{16}$$

The effective potential is therefore formally convex [16] as a function of the field ϕ_c.

IV. THE EFFECTIVE POTENTIAL FOR A FREE FIELD

In the case of a free scalar field, one may implement these definitions exactly. The unitary transformation

$$U(j) = \exp\left(-i \int (j/m^2)\pi(x) d^3x\right) \tag{17}$$

displaces the field $\phi(x)$,

$$U^\dagger \phi(x) U = \phi(x) + \frac{j}{m^2}\ , \tag{18}$$

and so relates the perturbed hamiltonian to the unperturbed one:

$$U^\dagger H U = H + \int j\,\phi(x)\, d^3x + \frac{L^3 j^2}{2m^2}. \tag{19}$$

Thus since traces are invariant under unitary transformations, the Helmholtz potential $A(j,T)$ for the free field

$$\begin{aligned}\exp\left[-\beta L^3\left(A + \frac{j^2}{2m^2}\right)\right] &= \mathrm{Tr}\,\exp\left[-\beta\left(H + \int j\,\phi\, d^3x + \frac{L^3 j^2}{2m^2}\right)\right] \\ &= \mathrm{Tr}\, U^\dagger e^{-\beta H} U \\ &= \mathrm{Tr}\, e^{-\beta H} = Z(\beta)\end{aligned} \tag{20}$$

is related to the logarithm (6) of the partition function of the unperturbed system by the equation

$$A(j,T) = -\frac{\ln Z(\beta)}{\beta L^3} - \frac{j^2}{2m^2}. \tag{21}$$

The effect of the linear perturbation $j \int \phi(x) d^3x$ is to displace the field by j/m^2, as shown by Eq. (18). So the mean value ϕ_c is

$$\phi_c = \frac{\partial A(j,T)}{\partial j} = -\frac{j}{m^2}. \tag{22}$$

One may also evaluate ϕ_c directly. Since the mean value $\langle \phi \rangle$ for the unperturbed theory is $\phi_c(0,T) = 0$, that of the perturbed theory is simply

$$\begin{aligned}\phi_c(j,T) &= \frac{\mathrm{Tr}\,\phi(x) U^\dagger e^{-\beta H} U}{\mathrm{Tr}\, U^\dagger e^{-\beta H} U} \\ &= \frac{\mathrm{Tr}\, U\phi(x) U^\dagger e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} \\ &= -\frac{j}{m^2}.\end{aligned} \tag{23}$$

It follows now from the definition (11), from Eqs. (21)–(23), and from the formula (6) for the partition function $Z(\beta)$ that the exact finite-temperature effective potential for the free scalar field of mass m is

$$\begin{aligned}V(\phi_c,T) &= A(j,T) - j\phi_c \\ &= \tfrac{1}{2} m^2 \phi_c^2 - \frac{\ln Z(\beta)}{\beta L^3} \\ &= \tfrac{1}{2} m^2 \phi_c^2 + \int \frac{d^3k}{(2\pi)^3}\left[\frac{\omega_k}{2} + \frac{\ln\left(1 - e^{-\beta\omega_k}\right)}{\beta}\right]\end{aligned} \tag{24}$$

with $\omega_k = \sqrt{k^2 + m^2}$. The effective potential $V(\phi_c,T)$ is real and convex. At its absolute minimum $\phi_c = 0$, the external current $j = -m^2\phi_c$ vanishes. In the limit $\beta \to \infty$, the potential $V(\phi_c,T)$ becomes the exact zero-temperature effective potential

$$V(\phi_c,0) = \tfrac{1}{2} m^2 \phi_c^2 + \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2}. \tag{25}$$

V. THE ONE-LOOP EFFECTIVE POTENTIAL

For a scalar field interacting with itself through a potential $V(\phi)$, the effect of the perturbing current j is to replace $V(\phi)$ by

$$V_j(\phi) = V(\phi) + j\phi. \tag{26}$$

The absolute minimum $\bar{\phi}$ of this altered potential is a root of the equation

$$0 = \frac{\partial V_j(\phi)}{\partial \phi} = \frac{\partial V(\phi)}{\partial \phi} + j. \tag{27}$$

To obtain the one-loop approximation to the Helmholtz potential, we replace the altered potential $V_j(\phi)$ in the definition (7) of $A(j,T)$ by the first terms of its Taylor-series expansion about the absolute minimum $\bar{\phi}$:

$$V_j(\phi) \approx V_j(\bar{\phi}) + \tfrac{1}{2}\frac{\partial^2 V_j(\bar{\phi})}{\partial \phi^2}(\phi - \bar{\phi})^2. \tag{28}$$

To zeroth order in $\hbar$, the minimum $\bar{\phi}$ is the mean value ϕ_c of the scalar field ϕ as defined by Eq. (9). The truncated series (28),

$$V_j(\phi) \approx V(\bar{\phi}) + j\bar{\phi} + \tfrac{1}{2}V''(\bar{\phi})\,(\phi - \bar{\phi})^2 \ , \tag{29}$$

describes a free scalar field of mass

$$m = \sqrt{V''(\bar{\phi})}. \tag{30}$$

The quantity $V''(\bar{\phi})$ is positive because $\bar{\phi}$ is a minimum of $V_j(\phi)$.

One may now express the Helmholtz potential $A(j,T)$ in terms of the kinetic energy $K = \int (1/2)\pi(x)^2 d^3x$ as

$$\begin{aligned}\exp[-\beta L^3 A(j,T)] &= \mathrm{Tr}\ \exp\left[-\beta\left(H + j\int\phi(x)d^3x\right)\right] \\ &= \mathrm{Tr}\ \exp\left[-\beta\left(K + \int V_j(x)d^3x\right)\right] \\ &\approx \exp\{-\beta L^3\left[V(\bar{\phi}) + j\bar{\phi}\right]\}\mathrm{Tr}\ \exp\left[-\beta\left(K + \int(1/2)m^2[\phi(x) - \bar{\phi}]^2 d^3x\right)\right] .\end{aligned} \tag{31}$$

So by using the unitary operator

$$U = \exp\left(i\int \bar{\phi}\pi(x)d^3x\right) \ , \tag{32}$$

which displaces the field $\phi(x)$ to $U^\dagger\phi(x)U = \phi(x) - \bar{\phi}$, one may write $A(j,T)$ approximately as

$$\begin{aligned} e^{-\beta L^3 A(j,T)} &\approx e^{-\beta L^3[V(\bar{\phi})+j\bar{\phi}]}\ \mathrm{Tr}\,[U^\dagger e^{-\beta H_0}U] \\ &\approx e^{-\beta L^3[V(\bar{\phi})+j\bar{\phi}]}\ \mathrm{Tr}\,[e^{-\beta H_0}] \\ &\approx e^{-\beta L^3[V(\bar{\phi})+j\bar{\phi}]}Z(\beta) \ , \end{aligned} \tag{33}$$

where $Z(\beta)$ is the exact partition function (6) for the free scalar field of mass m. Thus at the one-loop level, the Helmholtz potential is

$$A_1(j,T) = V(\bar{\phi}) + j\bar{\phi} - \frac{\ln Z(\beta)}{\beta L^3}. \tag{34}$$

The mean value ϕ_c and the mean $\bar{\phi}$ differ only by terms of order $\hbar$, which are due to the quantum fluctuations induced by the kinetic energy K. Specifically it follows from Eqs. (10) and (27) that this difference is

$$\phi_c = \bar{\phi} - \frac{\partial}{\partial j}\frac{\ln Z(\beta)}{\beta L^3}. \tag{35}$$

Thus by the extremal condition (27), the altered potential changes only by quantities that are of second order in $\hbar$ as $\bar{\phi}$ is replaced by ϕ_c:

$$V(\bar{\phi}) + j\bar{\phi} = V(\phi_c) + j\phi_c + O(\hbar^2). \tag{36}$$

We may therefore write the Helmholtz potential to first order in $\hbar$ in terms of the mean value ϕ_c of the field ϕ,

$$A_1(j,T) = V(\phi_c) + j\phi_c - \frac{\ln Z(\beta)}{\beta L^3} \ , \tag{37}$$

in which we have also freely replaced $\bar{\phi}$ by ϕ_c in the logarithm of Z, which itself is of order $\hbar$. Now by performing the Legendre transform (11), we find that the effective potential is

$$\begin{aligned} V_1(\phi_c,T) &= V(\phi_c) - \frac{\ln Z(\beta)}{\beta L^3} \\ &= V(\phi_c) + \int\frac{d^3k}{(2\pi)^3}\left[\frac{\omega_k}{2} + \frac{\ln\left(1 - e^{-\beta\omega_k}\right)}{\beta}\right] \end{aligned} \tag{38}$$

with

$$\omega_k = \sqrt{k^2 + V''(\phi_c)}, \tag{39}$$

which is the usual result.

Classical potentials that induce spontaneous symmetry breaking have second derivatives that are negative between their inflection points. When the second derivative $V''(\phi_c)$ is negative, the frequency ω_k becomes complex for small enough k, and the loop expansion for the effective potential fails.

The preceding integral of ω_k over momentum $\vec{k}$ diverges. We may renormalize the effective potential by interpreting the classical potential $V(\phi)$ as containing counterterms $V_{\mathrm{ct}}(\phi)$ of order $\hbar$ that are the same form as the terms of $V(\phi)$, apart from a constant term. By introducing a cutoff Λ and performing the integration, we find for the Helmholtz potential the expression

$$\begin{aligned} A_1(j,T) = V(\phi_c) + j\phi_c &+ \frac{V''(\phi_c)^2}{64\pi^2}\left[\ln\frac{V''(\phi_c)}{\mu^2} + \frac{1}{2}\right] \\ &+ \frac{\Lambda^4}{16\pi^2} + \frac{\Lambda^2 V''(\phi_c)}{16\pi^2} + \frac{V''(\phi_c)^2}{64\pi^2}\ln\frac{\mu^2}{4\Lambda^2} \\ &+ V_{\mathrm{ct}}(\phi_c) + \frac{T^4}{2\pi^2}\int_0^\infty x^2\ln\left(1 - e^{-\rho(x)}\right)dx \ , \end{aligned} \tag{40}$$

in which the renormalization point μ is arbitrary and

$\rho(x) = \omega_k/T$ is the square root

$$\rho(x) = \sqrt{x^2 + V''(\phi_c)/T^2}. \tag{41}$$

Thus, if we choose the counterterms minimally so that at ϕ_c they are

$$V_{\rm ct}(\phi_c) = -\frac{\Lambda^4}{16\pi^2} - \frac{\Lambda^2 V''(\phi_c)}{16\pi^2} + \frac{V''(\phi_c)^2}{64\pi^2}\ln\frac{4\Lambda^2}{\mu^2}, \tag{42}$$

then we obtain for the renormalized Helmholtz potential the formula

$$A_1(j,T) = V(\phi_c) + j\phi_c + \frac{V''(\phi_c)^2}{64\pi^2}\left[\ln\frac{V''(\phi_c)}{\mu^2} + \frac{1}{2}\right] + \frac{T^4}{2\pi^2}\int_0^\infty x^2 \ln\left(1 - e^{-\rho(x)}\right) dx, \tag{43}$$

in which because of Eq. (36) we may use either ϕ_c or $\bar\phi$ throughout. The effective potential is then given by the Legendre transform (11):

$$V_1(\phi_c,T) = A_1(j,T) - j\phi_c. \tag{44}$$

It will be useful in our discussion of the real effective potential to develop further the relation (35) between the mean value ϕ_c and the minimum $\bar\phi$ of the altered potential $V_j(\phi)$. By differentiating the extremal condition (27) with respect to j, we may find for the derivative of $\bar\phi$ the formula

$$\frac{\partial\bar\phi}{\partial j} = -\left[\frac{\partial^2 V(\bar\phi)}{\partial\bar\phi^2}\right]^{-1}. \tag{45}$$

It follows now from this formula and from Eqs. (10), (35), and (43) that the mean value ϕ_c is, to order $\hbar$,

$$\phi_c = \bar\phi - \frac{V'''(\bar\phi)}{32\pi^2}\left[\ln\frac{V''(\bar\phi)}{\mu^2} + 1\right] - \frac{T^2 V'''(\bar\phi)}{4\pi^2 V''(\bar\phi)}\int_0^\infty \frac{x^2 dx}{\rho(x)\left(e^{\rho(x)} - 1\right)}, \tag{46}$$

in which either ϕ_c or $\bar\phi$ may be used in the correction terms.

The archetypal example of a classical potential that exhibits spontaneous symmetry breaking is $V(\phi) = (\lambda/4)\left(\phi^2 - \sigma^2\right)^2$. This potential has a positive second derivative $V''(\phi) = \lambda\left(3\phi^2 - \sigma^2\right)$ only for fields $|\phi|$ that are greater than $\sigma/\sqrt{3}$. For smaller $|\phi_c|$, the one-loop effective potential $V_1(\phi_c,T)$ is complex. According to Eqs. (43) and (44), it is given by

$$V_1(\phi_c,T) = \frac{\lambda}{4}(\phi_c^2 - \sigma^2)^2 + \frac{\lambda^2(3\phi_c^2 - \sigma^2)^2}{64\pi^2}\left[\ln\left(\frac{\lambda(3\phi_c^2 - \sigma^2)}{\mu^2}\right) + \frac{1}{2}\right] + \frac{T^4}{2\pi^2}\int_0^\infty x^2\ln\left(1 - e^{-\rho(x)}\right) dx, \tag{47}$$

where now $\rho(x)$ is

$$\rho(x) = \sqrt{x^2 + \lambda(3\phi_c^2 - \sigma^2)/T^2} \tag{48}$$

and μ is an arbitrary renormalization mass. To this expression one may add arbitrary, finite counterterms of the form

$$\lambda^2(A\phi_c^4 + B\phi_c^2 + C) \tag{49}$$

from the renormalization of the hamiltonian H. Because of the first logarithm $\ln[\lambda(3\phi_c^2 - \sigma^2)/\mu^2]$, the effective potential $V_1(\phi_c,T)$ is complex for $|\phi_c| < \sigma/\sqrt{3}$, where it is not possible to quantize the approximate, altered theory.

The effective potential is the sum of a function of $\phi_c^2 - \sigma^2$ and a function of $3\phi_c^2 - \sigma^2$, and so cannot be convex. Its real part is concave, that is $\partial^2\Re V_1(\phi_c,T)/\partial\phi_c^2 < 0$, for most of the interval $-\sigma/\sqrt{3} < \phi_c < \sigma/\sqrt{3}$ for small λ, low temperatures T, and reasonable renormalization.

For this example, the relation (46) between the mean value ϕ_c and the minimum $\bar\phi$ is

$$\phi_c = \bar\phi - \frac{3\lambda\bar\phi}{16\pi^2}\left[\ln\frac{\lambda(3\bar\phi^2 - \sigma^2)}{\mu^2} + 1\right] - \frac{3T^2\bar\phi}{2\pi^2(3\bar\phi^2 - \sigma^2)}\int_0^\infty \frac{x^2 dx}{\rho(x)\left(e^{\rho(x)} - 1\right)}, \tag{50}$$

where $\rho(x)$ is given by Eq. (48).

The effective potential is complex in the region $-\sigma/\sqrt{3} < \phi_c < \sigma/\sqrt{3}$ because the absolute minimum $\bar\phi$ of the altered potential $V_j(\phi) = \lambda(\phi^2 - \sigma^2)^2/4 + j\phi$ always lies in the outer region $|\phi_c| \geq \sigma$. To verify this fact, we first note that the current j and the global minimum $\bar\phi$ of the altered potential $V_j(\phi)$ have opposite signs. Thus we may cast the extremal condition (27) for the absolute minimum $\bar\phi$ in the form

$$\bar\phi^2 = \sigma^2 - j/(\lambda\bar\phi) \geq \sigma^2. \tag{51}$$

A similar problem occurs for the generic symmetry-breaking potential.

VI. A REAL EFFECTIVE POTENTIAL

The reason for the complexity of the effective potential in models exhibiting spontaneous symmetry breaking is that the potential of the perturbed theory has the same second derivative $V_j''(\phi)$ as the original potential $V(\phi)$. Somewhat in the spirit of the composite-operator technique [25, 26], we may change the curvature of $V(\phi)$ by defining a more general Helmholtz potential $A(j,T;P)$ in which the linear probe $j\phi$ is replaced by a quadratic polynomial $jP(\phi)$:

$$\exp[-\beta L^3 A(j,T;P)] = \mathrm{Tr}\,\exp\left[-\beta\left(H + \int jP(\phi)d^3x\right)\right]. \tag{52}$$

On the one hand, it is clear that by this device we have not introduced any new divergences into the theory. On the other hand, it is also clear that the polynomial $P(\phi)$ is itself singular and requires regularization.

Now the derivative of the Helmholtz potential $A(j,T;P)$ with respect to the external current j is the mean value P_c,

$$\frac{\partial A(j,T;P)}{\partial j} = P_c \ , \tag{53}$$

defined as the mean value $\langle P(\phi)\rangle$ of the quadratic form $P(\phi)$:

$$P_c(j,T) = \frac{\mathrm{Tr}\, P(\phi(x)) \exp\left[-\beta\left(H + \int jP(\phi)d^3x\right)\right]}{\mathrm{Tr}\, \exp\left[-\beta\left(H + \int jP(\phi)d^3x\right)\right]}. \tag{54}$$

The classical field ϕ_c is still the mean value $\langle\phi\rangle_j$ of the quantum field ϕ:

$$\phi_c(j,T;P) = \frac{\mathrm{Tr}\, \phi(x) \exp\left[-\beta\left(H + \int jP(\phi)d^3x\right)\right]}{\mathrm{Tr}\, \exp\left[-\beta\left(H + \int jP(\phi)d^3x\right)\right]}. \tag{55}$$

We may now define a real effective potential $V(\phi_c,T;P)$ as the Legendre transform of the Helmholtz potential $A(j,T;P)$:

$$\begin{aligned} V(\phi_c,T;P) &= A(j,T;P) - j\frac{\partial A(j,T;P)}{\partial j} \\ &= A(j,T;P) - jP_c. \end{aligned} \tag{56}$$

The variable that is conjugate to j is P_c; so strictly speaking we probably should write $V(P_c,T;P)$ rather than $V(\phi_c,T;P)$. But all the potentials considered in this paper are actually and primarily functions of the external source j. And for a given perturbative ground state, the relationship between the source j and the mean value ϕ_c is one to one. Thus one may regard these potentials as functions of ϕ_c, which is the physically more significant variable.

Like the conventional effective potential $V(\phi_c,T)$, the real effective potential at its minima describes the unperturbed system. For where the effective potential $V(\phi_c,T;P)$ is stationary, the external current j vanishes,

$$\begin{aligned} 0 &= \frac{\partial V(\phi_c,T;P)}{\partial \phi_c} \\ &= \frac{\partial j}{\partial \phi_c}\left(\frac{\partial A(j,T;P)}{\partial j} - P_c\right) - j\frac{\partial P_c}{\partial \phi_c} \\ &= -j\frac{\partial P_c}{\partial \phi_c} \ , \end{aligned} \tag{57}$$

unless P_c exceptionally should be independent of ϕ_c.

This effective potential $V(\phi_c,T;P)$ is real but not necessarily convex. Its second derivative contains two terms

$$\frac{\partial^2 V(\phi_c,T;P)}{\partial \phi_c^2} = -\frac{\partial j}{\partial \phi_c}\frac{\partial P_c}{\partial \phi_c} - j\frac{\partial^2 P_c}{\partial \phi_c^2} \tag{58}$$

and has no definite sign because the first term is typically positive, while the second is typically negative. The real effective potential $V(\phi_c,T;P)$ is, however, formally convex as a function of the mean value P_c.

We shall need the relation between the mean value ϕ_c and the external source j. To that end we introduce the further-generalized Helmholtz potential $A(j',j,T;P)$ defined by the relation

$$\begin{aligned} &\exp[-\beta L^3 A(j',j,T;P)] \\ &\qquad = \mathrm{Tr}\exp\left[-\beta\left(H + \int (j'\phi + jP)\ d^3x\right)\right]. \end{aligned} \tag{59}$$

Clearly when j' vanishes, the potential $A(j',j,T;P)$ reduces to the generalized Helmholtz potential:

$$A(0,j,T;P) = A(j,T;P). \tag{60}$$

The mean value ϕ_c for the real effective potential is thus the partial derivative

$$\phi_c(j,T;P) = \left.\frac{\partial A(j',j,T;P)}{\partial j'}\right|_{j'=0} \tag{61}$$

evaluated at $j' = 0$. On the other hand, the potential $A(j',j,T;P)$ is the usual Helmholtz potential $A(j',T)$ for a theory in which the classical potential $V(\phi)$ has been shifted to $V(\phi) + jP(\phi)$:

$$A(j',j,T;P) = A(j',T)_{V+jP}. \tag{62}$$

Thus, by differentiating that potential $A(j',T)_{V+jP}$ instead of $A(j',j,T;P)$ and by using Eq. (10), we find that for the real effective potential the mean value $\phi_c(j,T;P)$ is equal to the mean value $\phi_c(j',T)_{V+jP}$ associated with the usual effective potential at $j' = 0$ for a theory with a shifted potential $V + jP$:

$$\begin{aligned} \phi_c(j,T;P) &= \left.\frac{\partial A(j',j,T;P)}{\partial j'}\right|_{j'=0} \\ &= \left.\frac{\partial A(j',T)_{V+jP}}{\partial j'}\right|_{j'=0} = \phi_c(0,T)_{V+jP}. \end{aligned} \tag{63}$$

By combining Eqs. (60) and (62), we find that the generalized Helmholtz potential $A(j,T;P)$ is the usual Helmholtz potential $A(j',T)_{V+jP}$ for the shifted potential at vanishing j':

$$A(j,T;P) = A(0,T)_{V+jP}. \tag{64}$$

In supersymmetric theories it may be appropriate to further generalize the perturbation P to a polynomial in both Fermi and Bose fields.

VII. THE MEANING OF EFFECTIVE POTENTIALS

The meaning of an effective potential is clearest at zero temperature. Since the perturbed hamiltonian $H + \int jP(\phi)d^3x$ is hermitian, it has eigenstates $|j\rangle$ with energy E_j:

$$\left(H + \int jP(\phi)d^3x\right)|j\rangle = E_j|j\rangle. \tag{65}$$

Thus, in the limit $\beta \to \infty$, the Helmholtz potential $A(j,T;P)$ by its definition (52) becomes the energy density $A(j,0;P) = E_j/L^3$ of the eigenstate $|j\rangle$ of the altered hamiltonian $H + \int jP(\phi)d^3x$ with minimum energy E_j. And so by Eq. (56), the effective potential

$$V(\phi_c,0;P) = A(j,0;P) - jP_c \tag{66}$$

is the mean value of the hamiltonian density in this state $|j\rangle$:

$$V(\phi_c, 0; P) = \frac{\langle j|H|j\rangle}{L^3}. \tag{67}$$

And from Eq. (57), it follows that the effective potential $V(\phi_c, 0; P)$ at its absolute minimum is the energy density of the ground state $|0\rangle$ of the unperturbed theory, i.e., the energy density of the physical vacuum.

At finite temperatures the potential $A(j,T;P)$ is the Helmholtz free-energy density of the mixture

$$\rho(j) = \frac{\exp\left[-\beta\left(H + \int jP(\phi)d^3x\right)\right]}{\mathrm{Tr}\,\exp\left[-\beta\left(H + \int jP(\phi)d^3x\right)\right]} \tag{68}$$

associated with the altered hamiltonian density $[H + \int jP(\phi)d^3x]/L^3$. The finite-temperature, real effective potential $V(\phi_c, T; P)$,

$$\begin{aligned} V(\phi_c, T; P) &= A(j,T;P) - j\frac{\partial A(j,T;P)}{\partial j} \\ &= A(j,T;P) - jP_c\ , \end{aligned} \tag{69}$$

is the analogue of the Gibbs free-energy density of this mixture. By differentiating the definition (52) of the Helmholtz free-energy density $A(j,T;P)$ with respect to the temperature T, one may relate it to the perturbed energy density $u(j) = \mathrm{Tr}\,\{\rho(j)\left[H + \int jP(\phi)d^3x\right]\}/L^3$ and the entropy density $s = -\mathrm{Tr}\,\rho(j)\ln\rho(j)/L^3$ of the mixture (68) by the equation

$$A(j,T;P) = u(j) - Ts. \tag{70}$$

It follows then from the relation (69) that the real effective potential or Gibbs free-energy density $V(\phi_c, T; P)$ is related to the energy density $u = \mathrm{Tr}\,\rho(j)H/L^3$ and entropy density s of the mixture $\rho(j)$ by the simpler equation

$$V(\phi_c, T; P) = u - Ts. \tag{71}$$

Unfortunately the mixture $\rho(j)$ given by Eq. (68) coincides with the unperturbed physical mixture $\rho = e^{-\beta H}/\mathrm{Tr}\,e^{-\beta H}$ only at the minima of $V(\phi_c, T; P)$, where the source j vanishes.

From this discussion it is clear that the choice of the polynomial $P(\phi)$ influences the real effective potential $V(\phi_c, T; P)$ except at its various minima. In particular, if one uses a quadratic polynomial P rather than a linear one, then one can avoid spurious complexities.

VIII. THE ONE-LOOP, REAL EFFECTIVE POTENTIAL

This section is concerned with the calculation of the one-loop, real effective potential $V(\phi_c, T; P)$ for an arbitrary renormalizable classical potential $V(\phi)$. The computation will closely follow that of the usual effective potential.

If the curvature $V''(\phi)$ of the classical potential $V(\phi)$ is positive, then one may take $P(\phi) = \phi$, and the two effective potentials are identical. If the curvature $V''(\phi)$ of the classical potential $V(\phi)$ is negative for some range of ϕ, then the probe $P(\phi)$ should be quadratic. There are then two cases, according to whether the classical potential has one or two minima.

If the potential $V(\phi)$ has a single minimum which we may call ϕ_1, then we may take the probe $P(\phi)$ to be half the square of the distance from ϕ_1:

$$P(\phi) \equiv P_1(\phi) = \frac{1}{2}(\phi - \phi_1)^2\,. \tag{72}$$

If the classical potential $V(\phi)$ has two minima, which we may call ϕ_1 and ϕ_2, then $V(\phi)$ also has one local maximum ϕ_0 between them. In this case we may take the probe $P(\phi)$ to be half the square of the distance from the local maximum ϕ_0:

$$P(\phi) \equiv P_2(\phi) = \frac{1}{2}(\phi - \phi_0)^2\,. \tag{73}$$

In both cases the first step is to find the minima, $\bar\phi$ of the altered potential $V_j = V(\phi) + jP(\phi)$. These minima $\bar\phi$ are roots of the equation

$$0 = V'(\bar\phi) + jP'(\bar\phi). \tag{74}$$

Without any loss of generality, we may let the leading term of the potential $V(\phi)$ be $\lambda\phi^4/4$.

In the case of a unique minimum ϕ_1, the derivative $V'(\phi)$ then will be the product of the three factors,

$$V'(\phi) = \lambda(\phi - \phi_1)(\phi - z)(\phi - z^*)\,. \tag{75}$$

So, since the derivative of the probe $P_1(\phi)$ is

$$P_1'(\phi) = (\phi - \phi_1)\,, \tag{76}$$

the minima $\bar\phi$ of the altered potential $V_j(\phi)$ are given by the quadratic equation

$$j = -\frac{V'(\bar\phi)}{P'(\bar\phi)} = -\lambda\left(\bar\phi - z\right)\left(\bar\phi - z^*\right) \le 0. \tag{77}$$

We choose to compute the real effective potential about the root $\bar\phi$ that is the absolute minimum of the altered potential $V_j(\phi)$.

When the potential $V(\phi)$ has two minima, ϕ_1 and ϕ_2, its derivative $V'(\phi)$ is the product of three factors,

$$V'(\phi) = \lambda(\phi - \phi_0)(\phi - \phi_1)(\phi - \phi_2)\,. \tag{78}$$

So, since the derivative of the probe $P_2(\phi)$ is

$$P_2'(\phi) = (\phi - \phi_0)\,, \tag{79}$$

the minima $\bar\phi$ of the altered potential $V_j(\phi) = V(\phi) + jP_2(\phi)$ are given by the quadratic equation

$$j = -\frac{V'(\bar\phi)}{P_2'(\bar\phi)} = -\lambda\left(\bar\phi - \phi_1\right)\left(\bar\phi - \phi_2\right). \tag{80}$$

The current j is positive for $\phi_1 < \bar\phi < \phi_2$. We choose to compute the real effective potential about the root $\bar\phi$ that is the absolute minimum of the altered potential $V(\phi)$.

By its definition (52), the Helmoltz potential $A_0(j,T;P)$ in both cases and to lowest order in $\hbar$ is

$$A_0(j,T;P) = V(\bar\phi) + jP(\bar\phi). \tag{81}$$

To find the order-$\hbar$ correction to this result, we replace the altered potential $V + jP$ in the definition (52) of the real Helmoltz potential $A(j, T; P)$ by its truncated Taylor series

$$V(\phi) + jP(\phi) \approx V(\bar{\phi}) + jP(\bar{\phi}) + \frac{1}{2}[V''(\bar{\phi}) + jP''(\bar{\phi})](\phi - \bar{\phi})^2 , \qquad (82)$$

and thereby reduce the problem to one that we have already solved (34). Thus, including counterterms, we find

$$A_1(j, T; P) = V(\bar{\phi}) + V_{\rm ct}(\bar{\phi}) + j\left[P(\bar{\phi}) + P_{\rm ct}(\bar{\phi})\right] - \frac{\ln Z(\beta)}{\beta L^3}. \qquad (83)$$

Here $V_{\rm ct}(\bar{\phi})$ is the quartic polynomial (42) of counterterms that we used to renormalize the ordinary effective potential; $P_{\rm ct}(\phi)$ is a quadratic polynomial in ϕ that we shall use to regularize the singular polynomial $P(\phi)$; and $P_{\rm ct}(\bar{\phi})$ is that polynomial with the field ϕ replaced by its mean value $\bar{\phi}$. The squared mass in $Z(\beta)$ is positive and writable simply as

$$m^2 = V''(\bar{\phi}) + jP''(\bar{\phi}) = V''(\bar{\phi}) + j \geq 0 \qquad (84)$$

because $\bar{\phi}$ is a minimum of $V_j(\phi)$ and because $P''(\bar{\phi}) = 1$ by construction [(72) and (73)]. Thus the Helmholtz potential is

$$A_1(j, T; P) = V(\bar{\phi}) + V_{\rm ct}(\bar{\phi}) + j\left[P(\bar{\phi}) + P_{\rm ct}(\bar{\phi})\right] + \int \frac{d^3k}{(2\pi)^3}\left[\frac{\omega_k}{2} + \frac{\ln\left(1 - e^{-\beta\omega_k}\right)}{\beta}\right] , \qquad (85)$$

where ω_k is now the real energy

$$\omega_k = \sqrt{k^2 + V''(\bar{\phi}) + j}. \qquad (86)$$

If we again introduce a cutoff Λ and perform the integration, then we find

$$A_1(j, T; P) = V(\bar{\phi}) + jP(\bar{\phi}) + \frac{[V''(\bar{\phi}) + j]^2}{64\pi^2}\left[\ln\frac{V''(\bar{\phi}) + j}{\mu^2} + \frac{1}{2}\right] + \frac{\Lambda^4}{16\pi^2} + \frac{\Lambda^2[V''(\bar{\phi}) + j]}{16\pi^2} + \frac{[V''(\bar{\phi}) + j]^2}{64\pi^2}\ln\frac{\mu^2}{4\Lambda^2} + V_{\rm ct}(\bar{\phi}) + jP_{\rm ct}(\bar{\phi}) + \frac{T^4}{2\pi^2}\int_0^\infty x^2 \ln\left(1 - e^{\rho(x)}\right) dx , \qquad (87)$$

in which μ is the renormalization point and $\rho(x)$ is the square root

$$\rho(x) = \sqrt{x^2 + \left[V''(\bar{\phi}) + j\right]/T^2}. \qquad (88)$$

Using the same counterterms $V_{\rm ct}(\bar{\phi})$ as the ones (42) that we used for the usual effective potential, we obtain

$$A_1(j, T; P) = V(\bar{\phi}) + jP(\bar{\phi}) + \frac{[V''(\bar{\phi}) + j]^2}{64\pi^2}\left[\ln\frac{V''(\bar{\phi}) + j}{\mu^2} + \frac{1}{2}\right] + \frac{\Lambda^2 j}{16\pi^2} + \frac{2jV''(\bar{\phi}) + j^2}{64\pi^2}\ln\frac{\mu^2}{4\Lambda^2} + jP_{\rm ct}(\bar{\phi}) + \frac{T^4}{2\pi^2}\int_0^\infty x^2 \ln\left(1 - e^{\rho(x)}\right) dx. \qquad (89)$$

All the terms on the right-hand side of this equation for $A_1(j, T; P)$ are functions of j, T, and the parameters of the theory. The same is true of $\bar{\phi}$. In particular the minimal choice of probe counterterms $P_{\rm ct}(\phi)$ evaluated at $\bar{\phi}$ is

$$P_{\rm ct}(\bar{\phi}) = -\frac{\Lambda^2}{16\pi^2} + \frac{2V''(\bar{\phi}) + j}{64\pi^2}\ln\frac{4\Lambda^2}{\mu^2} \qquad (90)$$

when expressed as a mixed function of j, $\bar{\phi}$, and the parameters of the theory. We shall now use the relations (77) and (80) that link j and $\bar{\phi}$ to write the probe counterterms $P_{\rm ct}(\bar{\phi})$ as functions of $\bar{\phi}$ and the parameters of the theory without j. The probe counterterms $P_{\rm ct}(\phi)$ will then be apparent.

In the case in which the classical potential $V(\phi)$ possesses a unique minimum ϕ_1, we may identify the counterterms $P_{1,\rm ct}(\phi)$ associated with the quadratic probe $P_1(\phi)$ by using the relation (77) between j and $\bar{\phi}$ to write the coefficient j of the logarithmically divergent term in (90) as $j = -\lambda\left|\bar{\phi} - z\right|^2$. The minimal choice of probe counterterms $P_{1,\rm ct}(\phi)$ then is

$$P_{1,\rm ct}(\phi) = -\frac{\Lambda^2}{16\pi^2} + \frac{2V''(\phi) - \lambda\left|\phi - z\right|^2}{64\pi^2}\ln\frac{4\Lambda^2}{\mu^2}. \qquad (91)$$

These probe counterterms $P_{1,\rm ct}(\phi)$ are of the same form as the probe $P_1(\phi)$, to wit, a quadratic polynomial in the variable ϕ.

In the case in which the classical potential $V(\phi)$ possesses two minima, ϕ_1 and ϕ_2, we exploit the relationship (80) between j and $\bar{\phi}$ to write the same coefficient j of the logarithmically divergent term in (90) as $j = -\lambda\left(\bar{\phi} - \phi_1\right)\left(\bar{\phi} - \phi_2\right)$. The minimal choice of probe counterterms $P_{2,\rm ct}(\phi)$ then is

$$P_{2,\rm ct}(\phi) = -\frac{\Lambda^2}{16\pi^2} + \frac{2V''(\phi) - \lambda\left(\phi - \phi_1\right)\left(\phi - \phi_2\right)}{64\pi^2}\ln\frac{4\Lambda^2}{\mu^2}. \qquad (92)$$

Like the probe $P_2(\phi)$, these counterterms $P_{2,\rm ct}(\phi)$ form a quadratic polynomial in the field ϕ.

We have seen that for an arbitrary renormalizable potential $V(\phi)$ with the same counterterms $V_{\rm ct}(\phi)$ as in the case of the usual effective potential, we may choose a polynomial $P(\phi)$ and probe counterterms $P_{ct}(\phi)$ of the same form as $P(\phi)$ so as to renormalize the real effective potential $V_1(\phi_c, T; P)$. If the potential $V(\phi)$ is of strictly non-negative curvature $V''(\phi) \geq 0$, then we may take $P(\phi) = \phi$ and $P_{ct}(\phi) = 0$, in which case the real effective potential $V(\phi_c, T; P)$ and the usual effective potential $V(\phi_c, T)$ coincide. If the potential $V(\phi)$ has a single minimum at $\phi = \phi_1$ with $V''(\phi) < 0$ for some ϕ, then we may take $P(\phi) = (\phi - \phi_1)^2/2$ as in Eq. (72). If the potential $V(\phi)$ has two minima separated by a local maximum at ϕ_0 with $V''(\phi) < 0$ for some ϕ, then we may take $P(\phi) = (\phi - \phi_0)^2/2$ as in Eq. (73). For these specific choices (72) and (73) of the probe polynomials $P_1(\phi)$ and $P_2(\phi)$, the probe counterterms $P_{1,ct}(\phi)$ and $P_{2,ct}(\phi)$ are of the same form as the probe polynomials themselves. For other choices of the probe polynomial, it may not be possible to transform the probe counterterms (90) so that they are of the same form as the probe polynomial; for such arbitrary choices of the probe polynomial, the real effective potential $V_1(\phi_c, T; P)$ may not be renormalizable.

With the counterterms $P_{ct}(\phi)$, the Helmholtz potential is now

$$\begin{aligned} A_1(j, T; P) = {} & V(\bar{\phi}) + jP(\bar{\phi}) \\ & + \frac{[V''(\bar{\phi}) + j]^2}{64\pi^2}\left[\ln\frac{V''(\bar{\phi}) + j}{\mu^2} + \frac{1}{2}\right] \\ & + \frac{T^4}{2\pi^2}\int_0^\infty x^2 \ln\left(1 - e^{\rho(x)}\right) dx. \end{aligned} \quad (93)$$

This formula and its counterpart (43) with ϕ_c replaced by $\bar{\phi}$ are an example of the relation (64) between the generalized Helmholtz potential $A(j, T; P)$ and the usual Helmholtz potential $A(j, T)_{V+jP}$ for the theory with shifted potential $V + jP$.

To compute the real effective potential $V_1(\phi_c, T; P)$, one must exploit the relationship (77) or (80) between $\bar{\phi}$ and j and use Eqs. (46) and (63) to relate the mean value ϕ_c to the minimum $\bar{\phi}$:

$$\begin{aligned} \phi_c = {} & \bar{\phi} - \frac{V'''(\bar{\phi})}{32\pi^2}\left[\ln\frac{V''(\bar{\phi}) + j}{\mu^2} + 1\right] \\ & - \frac{T^2 V'''(\bar{\phi})}{4\pi^2\left[V''(\bar{\phi}) + j\right]}\int_0^\infty \frac{x^2 dx}{\rho(x)\left(e^{\rho(x)} - 1\right)}\ , \end{aligned} \quad (94)$$

in which either ϕ_c or $\bar{\phi}$ may be used in the correction terms and $\rho(x)$ is the square root (88). By again using the relationship (77) or (80) between $\bar{\phi}$ and j, one finally may perform the Legendre transform (56) and compute the real effective potential $V_1(\phi_c, T; P)$. We shall carry out this procedure explicitly for the classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$.

IX. AN EXAMPLE

This section contains a detailed computation of the real effective potential for the case of the classical potential

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \sigma^2\right)^2. \quad (95)$$

We have seen in Sec. V that if we use the linear polynomial $P(\phi) = \phi$, then the resulting effective potential is complex for $|\phi_c| < \sigma/\sqrt{3}$. We shall find that by following Itzykson and Drouffe [28] and using the quadratic form $P(\phi) = \phi^2/2$, we may explore the whole region $|\phi_c| \leq \sigma$ with an effective potential $V(\phi_c, T; P)$ that remains real.

Since the potential (95) has indefinite curvature and two minima, the computation will follow the second of the two cases discussed in Sec. VIII. The minima are

$$\phi_1 = -\sigma \quad \text{and} \quad \phi_2 = \sigma; \quad (96)$$

they are separated by a local maximum at $\phi = \phi_0 = 0$. So the probe is

$$P(\phi) = \frac{1}{2}\phi^2. \quad (97)$$

The minima $\bar{\phi}$ of the altered potential

$$V_j(\phi) = V(\phi) + \frac{j}{2}\phi^2 \quad (98)$$

are the roots of the quadratic equation

$$j = -\lambda(\bar{\phi}^2 - \sigma^2) \quad (99)$$

or

$$\bar{\phi}^2 = \sigma^2 - j/\lambda. \quad (100)$$

Let us choose to quantize about the positive root

$$\bar{\phi} = \bar{\phi}_+ = +\sqrt{\sigma^2 - (j/\lambda)}. \quad (101)$$

The mass associated with $\bar{\phi}$ is given by

$$m^2 = V_j''(\bar{\phi}) = 2\lambda\bar{\phi}^2 \geq 0. \quad (102)$$

Since the current j is related to the minimum $\bar{\phi}$ by (99), we may write this squared mass also as

$$m^2 = 2\lambda\sigma^2 - 2j. \quad (103)$$

To lowest order in $\hbar$, the Helmholtz potential $A_0(j, T; P)$ is

$$A_0(j, T; P) = V(\bar{\phi}) + j\frac{\bar{\phi}^2}{2}\ , \quad (104)$$

in which $\bar{\phi}$ and j are related by $j = \lambda(\sigma^2 - \bar{\phi}^2)$. To this order the mean value ϕ_c and the minimum $\bar{\phi}$ are equal, and so the real effective potential $V_0(\phi_c, T; P)$ is just the classical potential $V(\phi_c)$:

$$\begin{aligned} V_0(\phi_c, T; P) &= A_0(j, T; P) - j\frac{\partial A(j, T; P)}{\partial j} \\ &= \frac{\lambda}{4}(\phi_c^2 - \sigma^2)^2. \end{aligned} \quad (105)$$

At the one-loop level, one finds by using Eqs. (96) and (102) that the regularizing counterterms (92) evaluated at $\bar{\phi}$ are

$$P_{ct}(\bar{\phi}) = -\frac{\Lambda^2}{16\pi^2} + \frac{\lambda(5\bar{\phi}^2 - \sigma^2)}{64\pi^2}\ln\frac{4\Lambda^2}{\mu^2}. \quad (106)$$

As a function of the external source j, the generalized Helmholtz potential $A_1(j, T; P)$ is then given by Eq. (93) as

$$A_1(j,T;P) = \frac{j\sigma^2}{2} - \frac{j^2}{4\lambda} + \frac{(2\lambda\sigma^2-2j)^2}{64\pi^2}\left[\ln\frac{2\lambda\sigma^2-2j}{\mu^2}+\frac{1}{2}\right] + \frac{T^4}{2\pi^2}\int_0^\infty x^2\ln\left(1-e^{\rho(x)}\right)dx\ , \quad (107)$$

where now $\rho(x)$ is the square root

$$\rho(x) = \sqrt{x^2+(2\lambda\sigma^2-2j)/T^2} = \sqrt{x^2+2\lambda\bar{\phi}^2/T^2}. \quad (108)$$

The generalized effective potential is defined (56) as the Legendre transform

$$V_1(\phi_c,T;P) = A(j,T;P) - j\frac{\partial A(j,T;P)}{\partial j}. \quad (109)$$

By performing the indicated differentiation with respect to j and by then expressing j as $\lambda(\sigma^2-\bar{\phi}^2)$, we may write $V_1(\phi_c,T;P)$ as

$$V_1(\phi_c,T;P) = \frac{\lambda}{4}(\bar{\phi}^2-\sigma^2)^2 + \frac{\lambda^2\bar{\phi}^2}{32\pi^2}\left[(4\sigma^2-2\bar{\phi}^2)\ln\frac{2\lambda\bar{\phi}^2}{\mu^2}+4\sigma^2-3\bar{\phi}^2\right] + \frac{T^4}{2\pi^2}\int_0^\infty x^2\left[\ln\left(1-e^{-\rho(x)}\right)+\frac{\lambda(\sigma^2-\bar{\phi}^2)}{T^2\rho(x)\left(e^{\rho(x)}-1\right)}\right]dx. \quad (110)$$

In the correction terms, which are of order $\hbar$, we may write indifferently $\bar{\phi}$ or ϕ_c. But in the first term $V(\bar{\phi})$ we must use Eq. (94) to distinguish ϕ_c from $\bar{\phi}$:

$$\phi_c = \bar{\phi} - \frac{3\lambda\bar{\phi}}{16\pi^2}\left[\ln\left(\frac{2\lambda\bar{\phi}^2}{\mu^2}\right)+1\right] - \frac{3T^2}{4\pi^2\bar{\phi}}\int_0^\infty\frac{x^2dx}{\rho(x)\left(e^{\rho(x)}-1\right)}. \quad (111)$$

Since ϕ_c and $\bar{\phi}$ differ by terms of order $\hbar$, we need keep only the leading term

$$V(\bar{\phi}) \approx V(\phi_c) + \lambda\phi_c\left(\phi_c^2-\sigma^2\right)\left(\bar{\phi}-\phi_c\right) \quad (112)$$

and may switch now to the variable ϕ_c throughout in the resulting formula for the one-loop, finite-temperature, real effective potential:

$$V_1(\phi_c,T;P) = \frac{\lambda}{4}(\phi_c^2-\sigma^2)^2 + \frac{\lambda^2\phi_c^2}{32\pi^2}\left[(4\phi_c^2-2\sigma^2)\ln\frac{2\lambda\phi_c^2}{\mu^2}+3\phi_c^2-2\sigma^2\right] + \frac{T^4}{2\pi^2}\int_0^\infty x^2\left[\ln\left(1-e^{-\rho(x)}\right)+\frac{\lambda(\phi_c^2-\sigma^2)}{2T^2\rho(x)\left(e^{\rho(x)}-1\right)}\right]dx\ , \quad (113)$$

in which $\rho(x)$ is the square root

$$\rho(x) = \sqrt{x^2+2\lambda\phi_c^2/T^2}. \quad (114)$$

One may adopt a specific set of renormalization conditions by adding finite counterterms to the preceding formula. A sensible set of conditions is $V_1(\sigma,0;P)=0$, $V_1'(\sigma,0;P)=0$, and $V_1''(\sigma,0;P)=m_H^2=2\lambda\sigma^2$. We may satisfy these conditions by adding the quartic polynomial

$$\frac{\lambda^2}{32\pi^2}\left(-8\phi^4+10\sigma^2\phi^2-3\sigma^4\right) \quad (115)$$

to the real effective potential (113) and setting $\mu^2=2\lambda\sigma^2$. The resulting expression is

$$V_1(\phi_c,T;P) = \frac{\lambda}{4}(\phi_c^2-\sigma^2)^2 + \frac{\lambda^2}{32\pi^2}\left[(4\phi_c^4-2\sigma^2\phi_c^2)\ln\frac{\phi_c^2}{\sigma^2}-5\phi_c^4+8\sigma^2\phi_c^2-3\sigma^4\right] + \frac{T^4}{2\pi^2}\int_0^\infty x^2\left[\ln\left(1-e^{-\rho(x)}\right)+\frac{\lambda(\phi_c^2-\sigma^2)}{2T^2\rho(x)\left(e^{\rho(x)}-1\right)}\right]dx \quad (116)$$

with $\rho(x)$ given by Eq. (114). If we set $\mu^2=2\lambda\sigma^2$ in the usual effective potential (47) and add to it these same counterterms, then it becomes

$$V_1(\phi_c,T) = \frac{\lambda}{4}(\phi_c^2-\sigma^2)^2 + \frac{\lambda^2}{64\pi^2}\left[(3\phi_c^2-\sigma^2)^2\ln\left(\frac{3\phi_c^2-\sigma^2}{2\sigma^2}\right)-\frac{23}{2}\phi_c^4+17\sigma^2\phi_c^2-\frac{11}{2}\sigma^4\right] + \frac{T^4}{2\pi^2}\int_0^\infty x^2\ln\left(1-e^{-\rho(x)}\right)dx\ , \quad (117)$$

where now $\rho(x)$ is

$$\rho(x) = \sqrt{x^2 + \lambda(3\phi_c^2 - \sigma^2)/T^2}. \tag{118}$$

Weinberg [13] found for the critical temperature T_C of this theory the approximate value $T_C \approx 2\,\sigma$ in the limit of weak coupling and high temperature. A numerical analysis of the formula (116) shows that the critical temperature runs from $T_C \approx 1.81\,\sigma$ for $\lambda = 0.1$ to $T_C \approx 1.74\,\sigma$ for $\lambda = 1$. At higher temperatures, the absolute minimum of $V_1(\phi_c, T; P)$ is at $\phi_c = 0$; at lower temperatures the absolute minimum lies in the region $\phi_c > 0.62\,\sigma$ for $\lambda = 0.1$ and in the region $\phi_c > 0.69\,\sigma$ for $\lambda = 1$. The transition is weakly first order because at $T = T_C$ the barrier separating the two minima is slight. This barrier disappears at the barrier temperature T_B, which eases from $T_B \approx 1.87\,\sigma$ for $\lambda = 0.1$ to $T_B \approx 1.865\,\sigma$ for $\lambda = 1$. For $T > T_B$, the field ϕ_c can roll classically from $\phi_c = \sigma$ to the absolute minimum at $\phi_c = 0$. The numbers quoted for $\lambda = 1$ may be unreliable, since two-loop effects can be significant at such strong coupling.

By differentiating Eq. (116) with respect to ϕ_c at $\phi_c = 0$, one may show that at all positive temperatures, the derivative of the real effective potential at $\phi_c = 0 + \epsilon$ is positive:

$$\left.\frac{\partial V_1(\phi_c, T; P)}{\partial \phi_c}\right|_{\phi_c = 0+\epsilon} > 0. \tag{119}$$

Thus the point $\phi_c = 0$ is a local minimum at all $T > 0$. The temperature at which this minimum disappears, which has been called T_2 in the literature [15], is therefore zero. In models of the early universe, inflation can occur if the field ϕ_c dawdles in this local minimum.

Figures 1–3 display in solid lines the real effective potential $V_1(\phi_c, T; P)$ from formula (116) at various temperatures. These figures also plot the real part of the usual effective potential $V_1(\phi_c, T)$ from formula (117) as dashes where its imaginary part vanishes ($\phi_c \geq \sigma/\sqrt{3}$) and as dots otherwise. The classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$ is plotted as short dashes in Fig. 1. The vertical axes are in units of $V(0) = \lambda\sigma^4/4$. The value $\lambda = 1$ was used in order to separate the curves. Figure 2 is an enlargement of Fig. 1 for $T = 0$ and σ. Figure 3 is an enlargement of Fig. 1 for the critical temperature $T_C \approx 1.74\,\sigma$ and the barrier temperature $T_B \approx 1.865\,\sigma$.

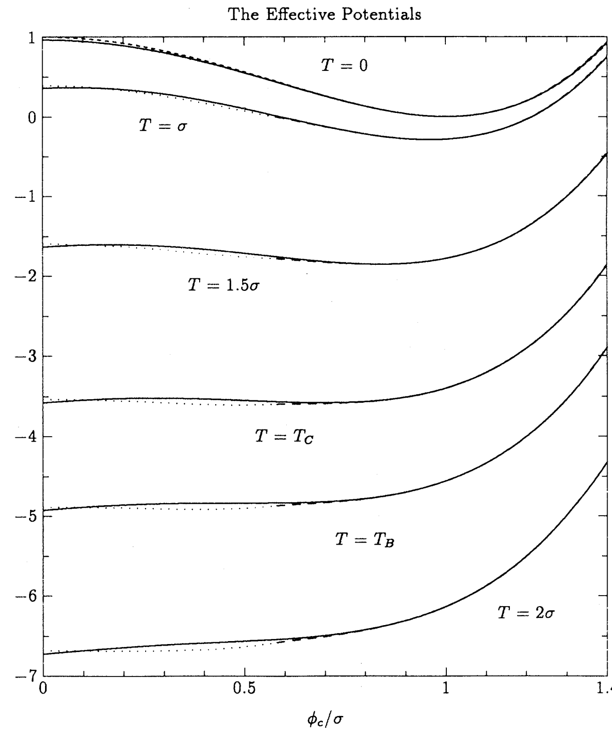

FIG. 1. For the classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$ with $\lambda = 1$ (short dashes), the one-loop, real effective potential $V_1(\phi_c, T; P)$ (solid lines) and the real part of the one-loop effective potential $V_1(\phi_c, T)$ [dashes for $\phi_c > \sigma/\sqrt{3}$, where $V_1(\phi_c, T)$ is real; dots otherwise] plotted in units of $V(0) = \lambda\sigma^4/4$ at various temperatures including the critical temperature $T_C \approx 1.74\,\sigma$ and the barrier temperature $T_B \approx 1.865\,\sigma$.

The real effective potential tracks the real part of the effective potential fairly closely at low temperatures and where the latter is real ($\phi_c \geq \sigma/\sqrt{3}$). The local minimum of the real effective potential $V_1(\phi_c, T; P)$ at $\phi_c = 0$ is clearly visible at $T \geq \sigma$ in Figs. 2 and 3.

By using the Haber-Weldon expansions of Bose-Einstein integrals [31], Weiler and I derived [30] as the high-temperature expansion of the real effective potential (116) the formula

$$V_1(\phi_c, T; P) = -\frac{\pi^2}{90}T^4 + \frac{\lambda}{24}\left(3\phi_c^2 - \sigma^2\right)T^2 + \frac{\lambda^{3/2}}{12\sqrt{2}\,\pi}\left(3\sigma^2|\phi_c| - 7|\phi_c|^3\right)T + \frac{\lambda^2}{16\pi^2}\left(2\phi_c^4 - \sigma^2\phi_c^2\right)\ln\frac{8\pi^2T^2}{\lambda\sigma^2}$$
$$+\frac{\lambda}{4}\left(\phi_c^2 - \sigma^2\right)^2 + \frac{\lambda^2}{32\pi^2}\left[-8\gamma\phi_c^4 + (4\gamma + 6)\sigma^2\phi_c^2 - 3\sigma^4\right] + O\left(\frac{\phi_c^2}{T^2}\right) \tag{120}$$

in which $\gamma \approx 0.577\,215\,66$. The derivation of this expansion is presented in the Appendix. The Dolan-Jackiw high-temperature expansion of the usual effective potential is [12]

$$V_1(\phi_c, T) = -\frac{\pi^2}{90}T^4 + \frac{\lambda}{24}\left(3\phi_c^2 - \sigma^2\right)T^2 - \frac{\lambda^{3/2}}{12\pi}\left(3\phi_c^2 - \sigma^2\right)^{3/2}T + \frac{\lambda^2}{64\pi^2}\left(3\phi_c^2 - \sigma^2\right)^2\ln\frac{T^2}{\lambda^2\left(3\phi_c^2 - \sigma^2\right)^2} + O(1). \tag{121}$$

Although the third and fourth terms of this formula for the usual effective potential $V_1(\phi_c, T)$ are imaginary for $|\phi_c| < \sigma/\sqrt{3}$, the first two terms, which are of order T^4 and T^2, do agree with the first two terms of the high-temperature expansion (120) of the real effective potential $V_1(\phi_c, T; P)$. Thus the seemingly outrageous approximation, often made in work on the early universe, of keeping only these two real terms of the high-temperature

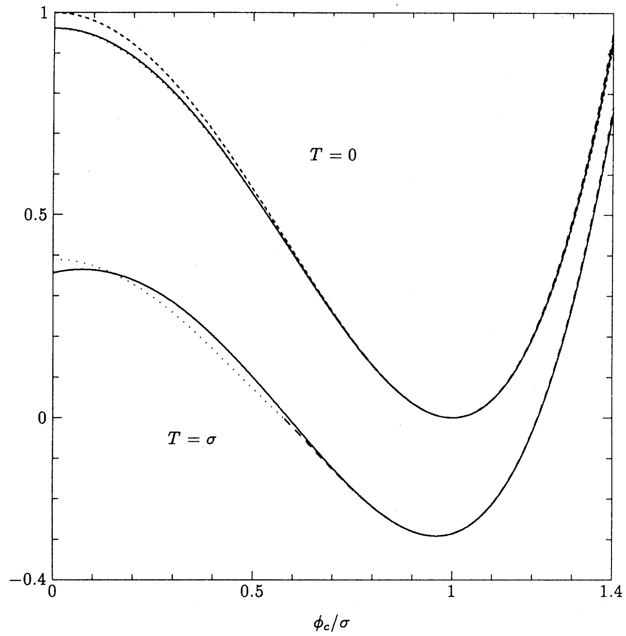

FIG. 2. For the classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$ with $\lambda = 1$, the one-loop, real effective potential $V_1(\phi_c, T; P)$ (solid lines) and the real part of the one-loop effective potential $V_1(\phi_c, T)$ [dashes for $\phi_c > \sigma/\sqrt{3}$, where $V_1(\phi_c, T)$ is real; dots otherwise] plotted in units of $V(0) = \lambda\sigma^4/4$ at temperatures $T = 0$ and $T = \sigma$.

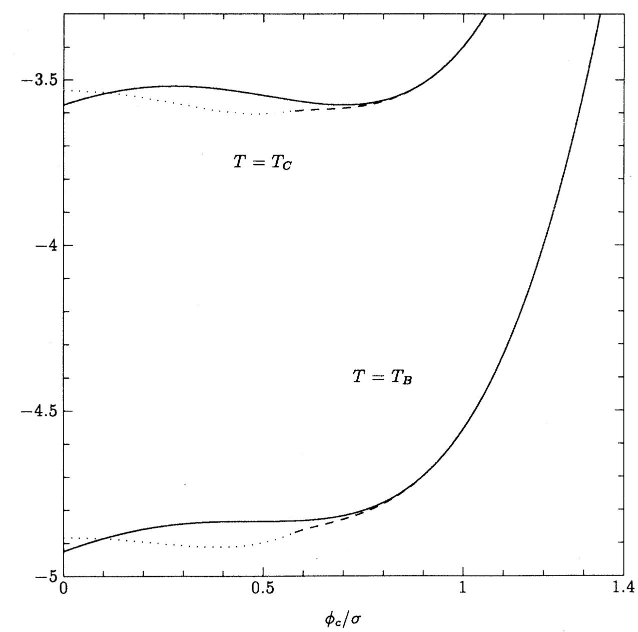

FIG. 3. For the classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$ with $\lambda = 1$, the one-loop, real effective potential $V_1(\phi_c, T; P)$ (solid lines) and the real part of the one-loop effective potential $V_1(\phi_c, T)$ [dashes for $\phi_c > \sigma/\sqrt{3}$, where $V_1(\phi_c, T)$ is real; dots otherwise] plotted in units of $V(0) = \lambda\sigma^4/4$ for the critical temperature $T_C \approx 1.74\,\sigma$ at which the minima at $\phi_c = 0$ and $\phi_c \approx 0.7\,\sigma$ are equally deep and for the barrier temperature $T_B \approx 1.865\,\sigma$, at which the barrier between these minima disappears.

expansion (121) turns out to be a reasonable one after all.

The third term in these expansions is real for $V_1(\phi_c, T; P)$ but imaginary for $V_1(\phi_c, T)$ when $|\phi_c| < \sigma/\sqrt{3}$. In the expansion of the real effective potential $V_1(\phi_c, T; P)$, the term $\lambda^{3/2}\sigma^2|\phi_c|T$ occurs with a positive coefficient and may have an astrophysical application [30]: it may trap the classical field ϕ_c at $\phi_c = 0$ long enough for cosmic inflation to ensue. This local minimum of the real effective potential $V_1(\phi_c, T; P)$ at $\phi_c = 0$ is far from the absolute $T = 0$ minimum at $\phi_c = \sigma$ and may be an artifact of the present method or of the one-loop approximation. But the contributions of fermions and gauge bosons to the one-loop effective potential have no terms linear in the product $|\phi_c|T$; they therefore cannot efface the local minimum of $V_1(\phi_c, T; P)$ at $\phi_c = 0$.

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APPENDIX

This appendix contains a derivation of the high-temperature expansion of the real effective potential (113) for the case of the classical potential $V(\phi) = (\lambda/4)(\phi^2 - \sigma^2)^2$. This derivation was originally suggested to me by Weiler and was carried out by him and me at Aspen.

The finite-temperature integral in the expression (113) is

$$I(T, y) = \frac{T^4}{2\pi^2} \int_0^\infty x^2 \left[\ln\left(1 - e^{-\sqrt{x^2+y^2}}\right) + \frac{\lambda(\phi_c^2 - \sigma^2)}{2T^2\sqrt{x^2+y^2}\left(e^{\sqrt{x^2+y^2}} - 1\right)} \right] dx \quad \text{(A1)}$$

in which

$$y^2 = 2\lambda\phi_c^2/T^2. \quad \text{(A2)}$$

This integral is finite for all temperatures T and vanishes at $T = 0$.

We may split the integral $I(T, y)$ into two parts:

$$I(T,y) = J_1(T,y) + J_2(T,y) \ , \tag{A3}$$

in which

$$J_1(T,y) = \frac{T^4}{2\pi^2}\int_0^\infty x^2 \ln\left(1 - e^{-\sqrt{x^2+y^2}}\right) \ , \tag{A4}$$

and

$$J_2(T,y) = \frac{T^2\lambda(\phi_c^2 - \sigma^2)}{4\pi^2}\int_0^\infty \frac{x^2}{\sqrt{x^2+y^2}}\frac{1}{\left(e^{\sqrt{x^2+y^2}} - 1\right)}. \tag{A5}$$

Integrals of these forms have been analyzed by Haber and Weldon [30]. By using Eqs. (2) and (3) of their paper, we see that the integrals J_1 and J_2 are given by

$$J_1(T,y) = -\frac{4T^4}{\pi^2}h_5(y,0) \tag{A6}$$

and

$$J_2(T,y) = \frac{T^2\lambda(\phi_c^2 - \sigma^2)}{2\pi^2}h_3(y,0). \tag{A7}$$

From Eq. (12) of their paper, we see that $h_n(y,0) = h_n^e(y,0)$. We may therefore use their high-temperature or small-y expansion (D1) of $h_{2l+1}^e(y,r)$ for the cases $l = 1$ and 2 and $r = 0$. The case $l = 1$ and $r = 0$ gives $h_3(y,0)$. Since ${}_3F_2(1,1,0;\frac{3}{2},2;0)$, ${}_2F_1(0,1;\frac{1}{2};0)$, and ${}_2F_1(-k,-1-k;\frac{1}{2};0)$ for $k > 0$ are all equal to unity, we may write their expansion (D1) as

$$\begin{aligned} h_3(y,0) = &-\frac{\pi y}{4} - \frac{y^2}{8}\left\{\ln\left(\frac{y}{4\pi}\right) + \frac{1}{2}[\gamma - \psi(2)]\right\} \\ &+ \frac{\zeta(2)}{2} - \frac{y^2}{8}\sum_{k=1}^{\infty}(-1)^k\left(\frac{y}{4\pi}\right)^{2k} \\ &\times\frac{\Gamma(2k+1)\zeta(2k+1)}{\Gamma(k+1)\Gamma(k+2)}. \end{aligned} \tag{A8}$$

We keep only terms of order y^2 or greater, so we may ignore the complicated sum. Thus, since $\psi(2) = 1 - \gamma$ and $\zeta(2) = \pi^2/6$, we find for $h_3(y,0)$ the expansion

$$h_3(y,0) = \frac{\pi^2}{12} - \frac{\pi y}{4} - \frac{y^2}{8}\left[\ln\left(\frac{y}{4\pi}\right) + \gamma - \frac{1}{2}\right] + O(y^4). \tag{A9}$$

The case $l = 2$ and $r = 0$ gives $h_5(y,0)$. Because ${}_3F_2(1,1,-1;\frac{3}{2},2;0)$ and ${}_2F_1(-k,2-k;\frac{1}{2};0)$ for $k = 0,1$ and ${}_2F_1(-k,-2-k;\frac{1}{2};0)$ for $k > 0$ are all equal to unity, we may write their formula (D1) for $h_5(y,0)$ as

$$\begin{aligned} h_5(y,0) = &\frac{\pi y^3}{48} + \frac{y^4}{128}\left\{\ln\left(\frac{y}{4\pi}\right) + \frac{1}{2}[\gamma - \psi(3)]\right\} \\ &+\frac{\zeta(4)}{4} - \frac{y^2\zeta(2)}{16} + \frac{y^4}{64}\sum_{k=1}^{\infty}(-1)^k\left(\frac{y}{4\pi}\right)^{2k} \\ &\times\frac{\Gamma(2k+1)\zeta(2k+1)}{\Gamma(k+1)\Gamma(k+2)}. \end{aligned} \tag{A10}$$

We keep only terms of order y^4 or greater, and so we may ignore the complicated sum. Thus, since $\psi(3) = \frac{3}{2} - \gamma$ and $\zeta(4) = \pi^4/90$, we find for $h_5(y,0)$ the expansion

$$\begin{aligned} h_5(y,0) = &\frac{\pi^4}{360} - \frac{\pi^2 y^2}{96} + \frac{\pi y^3}{48} \\ &+\frac{y^4}{128}\left[\ln\left(\frac{y}{4\pi}\right) + \gamma - \frac{3}{4}\right] + O(y^6). \end{aligned} \tag{A11}$$

Since the integral $I(T,y)$ is an even function of ϕ_c or of y, it follows that we must interpret the variable y as $y = \sqrt{2\lambda}|\phi_c|/T$. Thus, by combining Eqs. (A1), (A3), (A6), (A7), (A9), and (A11), we find for the integral $I(T,y)$ the high-temperature expansion

$$\begin{aligned} I(T,y) = &-\frac{\pi^2 T^4}{90} + \frac{T^2\lambda(3\phi_c^2 - \sigma^2)}{24} + \frac{T\lambda^{\frac{3}{2}}\sigma^2|\phi_c|}{4\pi\sqrt{2}} - \frac{7T\lambda^{\frac{3}{2}}|\phi_c|^3}{12\pi\sqrt{2}} \\ &+ \frac{\lambda^2\phi_c^2(2\phi_c^2 - \sigma^2)}{16\pi^2}\left[\ln\left(\frac{8\pi^2T^2}{\lambda\phi_c^2}\right) - 2\gamma\right] + \frac{5\lambda^2\phi_c^4}{32\pi^2} - \frac{\lambda^2\phi_c^2\sigma^2}{16\pi^2} + O\left(\frac{\phi_c^2}{T^2}\right). \end{aligned} \tag{A12}$$

Here $\gamma \equiv C \approx 0.577\,215\,665$ is the Euler-Mascheroni constant. If we substitute this formula for $I(T,y)$ into Eq. (116), then we obtain this high-temperature expansion of the real effective potential:

$$\begin{aligned} V_1(\phi_c,T;P) = &-\frac{\pi^2}{90}T^4 + \frac{\lambda}{24}\left(3\phi_c^2 - \sigma^2\right)T^2 + \frac{\lambda^{3/2}}{12\sqrt{2}\,\pi}\left(3\sigma^2|\phi_c| - 7|\phi_c|^3\right)T + \frac{\lambda^2}{16\pi^2}\left(2\phi_c^4 - \sigma^2\phi_c^2\right)\ln\frac{8\pi^2T^2}{\lambda\sigma^2} \\ &+\frac{\lambda}{4}\left(\phi_c^2 - \sigma^2\right)^2 + \frac{\lambda^2}{32\pi^2}\left[-8\gamma\phi_c^4 + (4\gamma+6)\sigma^2\phi_c^2 - 3\sigma^4\right] + O\left(\frac{\phi_c^2}{T^2}\right). \end{aligned} \tag{A13}$$

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