Unitary gauge theories of noncompact groups

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It is noted that the use of an internal metric field allows one to gauge noncompact internal-symmetry groups without sacrificing unitarity. The possibility that such theories could be rendered renormalizable is discussed.

In a recent interesting paper,[1] Hsu and Xin have shown that a specific gauge theory of the noncompact group SL$(2,C)$ is not unitary. The theory in question, which they abstracted from a work of Wu and Yang,[2] has a Hamiltonian that is not bounded below. To avoid this difficulty, they quantized some of the gauge mesons with a negative metric. They then showed that this negative metric leads to a loss of unitarity that cannot be repaired without sacrificing gauge invariance.

Our purpose here is to recall that unitary gauge theories of arbitrary noncompact groups have been constructed[3–5] and to suggest how they might be made renormalizable.

Let us suppose that we wish to construct a gauge theory of n complex scalar fields ϕ_i whose Lagrangian is invariant under the local action

$$\phi_i'(x)=a_{ij}(x)\phi_j(x) \tag{1}$$

of the general linear group[6] of all complex nonsingular $n\times n$ matrices, GL(n,C). The key step is to introduce a metric field $g_{ij}(x)$ that is Hermitian and positive[7] and that transforms as

$$g_{ij}'(x)=[a^{-1}(x)]^*_{ki}g_{kl}(x)[a^{-1}(x)]_{lj} \ . \tag{2}$$

It is convenient to use a notation in which ϕ is a column vector and g a matrix. Then a suitable covariant derivative for ϕ is

$$\phi_{;\mu}(x)=\phi(x)_{,\mu}-A_\mu(x)\phi(x) \ , \tag{3}$$

where $\phi_{,\mu}$ is the ordinary derivative of ϕ with respect to x^μ and A_μ is an $n\times n$ matrix of gauge fields that transforms as

$$A_\mu'(x)=a(x)A_\mu(x)a^{-1}(x)+a(x)_{,\mu}a^{-1}(x) \ . \tag{4}$$

The curvature tensor $F_{\mu\nu}$,

$$F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}+[A_\mu,A_\nu] \ , \tag{5}$$

transforms covariantly, $F_{\mu\nu}'=aF_{\mu\nu}a^{-1}$. The covariant derivative of g is

$$g_{;\mu}=g_{,\mu}+gA_\mu+A_\mu^\dagger g \ . \tag{6}$$

Similarly, the covariant derivatives of $\phi^{;\nu}$, $F^{\sigma\tau}$, and $g^{;\nu}$ are

$$\phi^{;\nu}_{;\mu}=\phi^{;\nu}_{,\mu}-A_\mu\phi^{;\nu} \ , \tag{7}$$

$$F^{\sigma\tau}_{;\mu}=F^{\sigma\tau}_{,\mu}+[F^{\sigma\tau},A_\mu] \ , \tag{8}$$

$$g^{;\nu}_{;\mu}=g^{;\nu}_{,\mu}+g^{;\nu}A_\mu+A_\mu^\dagger g^{;\nu} \ . \tag{9}$$

The Lagrange density

$$L=-\frac{1}{4e^2}\mathrm{tr}(F^\dagger_{\mu\nu}gF^{\mu\nu}g^{-1})+\frac{m^2}{4}\mathrm{tr}(g_{;\mu}g^{-1}g^{;\mu}g^{-1}) + \phi^\dagger_{;\mu}g\phi^{;\mu}-V(\phi^\dagger g\phi) \tag{10}$$

is invariant under the gauge transformation (1). The field equations are

$$\phi^{;\mu}_{;\mu}+g^{-1}g_{;\mu}\phi^{;\mu}+V'\phi=0 \ , \tag{11}$$

$$F^{\mu\nu}_{;\nu}=[F^{\mu\nu},g^{-1}g_{;\nu}]+e^2\phi^{;\mu}\phi^\dagger g-e^2m^2g^{-1}g^{;\mu} \ , \tag{12}$$

$$g^{;\mu}_{;\mu}=g_{;\mu}g^{-1}g^{;\mu}+(2e^2m^2)^{-1}g[g^{-1}F^\dagger_{\mu\nu}g,F^{\mu\nu}] +2m^{-2}g\phi^{;\mu}\phi^\dagger_{;\mu}g-2m^{-2}g\phi V'\phi^\dagger g \tag{13}$$

where V' is the derivative of V. The Hamiltonian is

$$H=\frac{1}{4e^2}\mathrm{tr}(F^\dagger_{\mu\nu}gF_{\mu\nu}g^{-1})+\frac{m^2}{4}\mathrm{tr}[(g_{;\mu}g^{-1})^2] + \phi^\dagger_{;\mu}g\phi_{;\mu}+V \ , \tag{14}$$

in which all Lorentz indices are down.

The Hamiltonian is gauge invariant. It is also positive as one may show by transforming[8] to the gauge $g(x)=1$. Since the Hamiltonian is automatically positive, we may quantize the theory (in, e.g., the temporal gauge $A_0=0$) without introducing any negative metrics. Thus unitarity follows from the natural positivity of the Hamiltonian H.

Unfortunately, renormalizability is less automatic. The presence of g^{-1} in L ruins renormalizability unless we quantize in, e.g., a gauge with $g(x)=1$. But in every gauge, the gauge mesons W_μ associated with the Hermitian part of A_μ acquire the mass $M=\sqrt{2}em$. The theory is thus presumably nonrenormalizable, unless we set $m=0$ and quantize it in a gauge with $g(x)=1$.

If $m=0$, then the field equation (13) for $g(x)$ becomes the constraint

$$0=\frac{1}{4e^2}[g^{-1}F^{\dagger}_{\mu\nu}g,F^{\mu\nu}]g^{-1}+\phi^{;\mu}\phi^{\dagger}_{;\mu}-\phi V'\phi^{\dagger} \ . \quad (15)$$

This constraint would severely hobble the theory if it were not for the sum over the Lorentz indices. In fact, one may show that this constraint actually follows from the other two field equations (11) and (12) when $m=0$. The Lagrangian (10), with $m=0$ and in the gauge $g(x)=1$, would seem by power counting to be renormalizable.[8,9] We hope to report whether this is actually true in a future publication.

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[2]T. T. Wu and C. N. Yang, Phys. Rev. D 13, 3233 (1976).
[3]K. Cahill, Phys. Rev. D 18, 2930 (1978); 20, 2636 (1979); J. Math. Phys. 21, 2676 (1980).
[4]J. E. Kim and A. Zee, Phys. Rev. D 21, 1939 (1980).
[5]B. Julia and J. F. Luciani, Phys. Lett. 90B, 270 (1980).
[6]One may obtain a unitary gauge theory of the group SL(2,C) by setting $n=2$ in Eqs. (1)–(14) and by making A_{μ} traceless.
[7]The positivity of the metric field $g(x)$ may be ensured by writing it in terms of a more fundamental field $h(x)$ as $g=h^{\dagger}h$.
[8]For gauge groups G smaller than GL(n,C) or GL(n,R), it is useful to restrict $g(x)$ to the quotient space, G/H, where H is the maximal compact subgroup of G, in order that the gauge $g(x)=1$ be available. The Hamiltonian is positive, however, even if g is not so restricted.
[9]The potential V is assumed to be at most a quadratic function of the quantity $\phi^{\dagger}g\phi$.