On the unification of the gravitational and electronuclear forces

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Theories are constructed in which the action is invariant under local transformations of the spin and species indices of the fields and under general coordinate transformations of the spacetime coordinates. Such theories eventually might be used to unify gravity with the strong and electroweak forces. Some examples are given.

I. INTRODUCTION

Gauge theory seems to describe well the four fundamental forces. Yet the kind of gauge theory that describes electromagnetism and the strong and weak nuclear forces differs from the kind that describes gravity. The gauge group of the electronuclear forces transforms species indices but not spin indices. The gauge group of gravity transforms spin indices but not species indices. The action of the electronuclear forces is invariant under global Poincaré transformations; the action of gravity is invariant under general coordinate transformations. In the present paper, theories are constructed in which the action is invariant under local transformations of the species and spin indices and under general coordinate transformations of the spacetime coordinates. The gauge group may be said to be collateral because it acts on spin and species indices in a parallel and even-handed fashion. Such theories eventually might be used to unify the electronuclear forces with gravity . Supergravity[1] is an example of a collateral gauge theory. The purpose here is to provide another example.

A collateral gauge transformation is both a local linear transformation of the fields ψ^a and a general coordinate transformation of the spacetime coordinates x^μ. It associates with each spacetime point x both an element $g(x)$ of its gauge group G and an image spacetime point x'. A collateral gauge theory may associate various representations T with its gauge group G. The representation T acts on the spin and species indices of the fields, changing a field of one type into one of a different type. Thus a collateral transformation changes the matter field $\psi^a(x)$ into

$$\psi^a(x)'=T^a{}_b[g(x)]\psi^b(x') \tag{1.1}$$

and subjects spacetime coordinates to the general coordinate transformation $x^\mu \rightarrow x'^\mu$. The indices a and b run over the species and spin indices of the field ψ.

Yang-Mills theories correspond to the choice G compact and T a unitary representation of G acting only on species indices. General relativity corresponds to the choice $G=\mathrm{SL}(2,C)$ and T a representation of SL(2,C) acting only on spin indices. There is considerable freedom in the choice of G and T. Once they are chosen, there is a wider variety of possible Lagrangians than in a Yang-Mills theory because there are more invariants that can be formed from a tetrad and a connection than from a connection alone. The class of collateral gauge theories is therefore very broad.

In theories having local or global SL(2,C) symmetry, there is a generic doubling of matter fields because the representations (j,j') and (j',j) are inequivalent. In a collateral gauge theory, T is typically larger than SL(2,C) and the representations T, T^*, T^{-1T}, and $T^{-1\dagger}$ often are all inequivalent. In collateral gauge theories therefore, there is a natural quadrupling of matter fields. Instead of having two Weyl spinors, which one may combine into one Dirac spinor, one has four Weyl spinors, which one may combine into two Dirac spinors. If T acts on a family of quark and lepton fields, then one may have four families. This quadrupling of matter fields may partially explain the apparent proliferation of quarks and leptons.

In a famous paper,[2] Coleman and Mandula showed that under certain assumptions the symmetries of the S matrix take the form of a direct product of the Poincaré group with an internal symmetry group. Collateral theories gauge more general groups. Thus those collateral gauge symmetries of the action that are not of the direct-product type may fail to be symmetries of the S matrix. In general the symmetries of the vacuum and of the S matrix are fewer than those of the action because the tetrads assume vacuum expected values, which determine the metric of spacetime.

Collateral covariant derivatives are defined in Sec. II. Fermions are discussed in Sec. III. Various actions for the gauge fields are described in Sec. IV. Some examples of collateral gauge theories are given in Sec. V. One of them attempts to unify gravity with the electronuclear forces in a toy model with two 16-member families of quarks and leptons. These examples illustrate how the vacuum expected values of the tetrads render the vacuum and the S matrix less symmetric than the action.

II. COVARIANT DERIVATIVES

For some purposes it is convenient to use a matrix notation in which the matrix $T[g(x)]$ is written as $T(x)$ and to imagine that a collateral gauge transformation is implemented by an operator U, which is a functional of the function $g(x)$ in G. In this notation, the matter field $\psi(x)$ is a vector. Under a collateral transformation, it is changed into

$$\psi(x)'=U^{-1}\psi(x)U=T(x)\psi(x') , \quad (2.1)$$

where x' is the image of x under the associated general coordinate transformation. Since U is independent of x, the derivative of ψ with respect to x^μ is transformed into

$$\begin{aligned}\partial_\mu\psi(x)'&=U^{-1}\partial_\mu\psi(x)U=\partial_\mu[U^{-1}\psi(x)U]\\&=\partial_\mu[T(x)\psi(x')]=\partial_\mu x'^\nu\partial'_\nu[T(x)\psi(x')] ,\end{aligned} \quad (2.2)$$

where use has been made of the chain rule and of the definition $\partial'_\nu=\partial/\partial x'^\nu$. A suitable covariant derivative is

$$D_\mu=\partial_\mu+A_\mu \quad (2.3)$$

provided the connection A_μ transforms as

$$A_\mu(x)'=\partial_\mu x'^\nu T(x)[A_\nu(x')+\partial'_\nu]T^{-1}(x) . \quad (2.4)$$

For then $D_\mu\psi$ transforms as

$$\begin{aligned}D_\mu\psi(x)'&=U^{-1}D_\mu\psi(x)U\\&=\partial_\mu x'^\nu T(x)[\partial'_\nu+A_\nu(x')]\psi(x')\\&=\partial_\mu x'^\nu T(x)D_\nu\psi(x') .\end{aligned} \quad (2.5)$$

The curvature tensor

$$F_{\mu\nu}=[D_\mu,D_\nu] \quad (2.6)$$

transforms as

$$F_{\mu\nu}(x)'=\partial_\mu x'^\sigma\partial_\nu x'^\lambda T(x)F_{\sigma\lambda}(x')T^{-1}(x) \quad (2.7)$$

since

$$\partial_\mu x'^\sigma\partial'_\sigma\partial_\nu x'^\lambda=\partial_\nu x'^\sigma\partial'_\sigma\partial_\mu x'^\lambda . \quad (2.8)$$

III. FERMIONS

Because T is in general larger than SL(2,C), it is necessary to generalize the tetrad and to allow for the possibility of four classes of fermions. It is an open question whether this generic quadrupling of kinds of fermions has any relation to the doubling of flavors that is exhibited by the known quarks and leptons or to the repetition of the quark-lepton families.

A suitable Lagrange density L_ψ for a spin-$\frac{1}{2}$ field ψ that transforms as

$$\psi(x)'=T(x)\psi(x') , \quad (3.1)$$

where x' is the image of x under the associated general coordinate transformation, may be constructed from the scalar I_ψ,

$$I_\psi=\frac{i}{2}\psi^\dagger e^\mu D_\mu\psi + \text{H.c.} \quad (3.2)$$

in which H.c. means Hermitian conjugate. The scalar I_ψ will be invariant if the tetrad e^μ transforms as

$$e^\mu(x)'=\partial'_\nu x^\mu T^{-1\dagger}(x)e^\nu(x')T^{-1}(x) . \quad (3.3)$$

(One may obtain L_ψ by multiplying I_ψ by a scale factor $\sqrt{-g}$ that, under a gauge transformation, is multiplied by the Jacobian of the general coordinate transformation $x\rightarrow x'$. The metric $g_{\mu\nu}$ is defined in Sec. IV.)

The tetrad e^μ may be expanded in terms of a suitable set of Hermitian orthonormal matrices t_i as

$$e^\mu(x)=e_i^\mu(x)t_i . \quad (3.4)$$

By substituting this expansion into the transformation law (3.4) and using the trace relation

$$\delta_{i,j}=tr(t_it_j) , \quad (3.5)$$

one may derive for the tetrad field e_i^μ its transformation rule

$$e_i^\mu(x)'=\partial'_\nu x^\mu W_{ij}(x)e_j^\nu(x') \tag{3.6}$$

in which the coefficients W_{ij} are given by the traces

$$W_{ij}(x)=tr[t_i T^{-1\dagger}(x)t_j T^{-1}(x)] . \tag{3.7}$$

The W's form a second-rank tensor representation of T. They are real, as are the tetrad fields e_i^μ.

The number of t_i's that are required depends upon the representation T. In the simplest case in which T is SL(2,C) there are four, which may be taken to be the unit matrix and the three Pauli matrices. Other examples are given in Sec. V.

When T and T^* are inequivalent, it is necessary to allow for the possibility that some spin-$\frac{1}{2}$ field χ might transform with T^*,

$$\chi(x)'=T^*(x)\chi(x') , \tag{3.8}$$

rather than with T. By examining the complex conjugates of the preceding equations for the field ψ, one may show that a suitable Lagrange density L_χ may be constructed from the scalar I_χ:

$$I_\chi=\frac{i}{2}\chi^\dagger e^{*\mu}D_\mu^*\chi+\text{H.c.} \tag{3.9}$$

When T, T^*, and $T^{-1\dagger}$ are all inequivalent, one may have a field φ that transforms as

$$\varphi(x)'=T^{-1\dagger}(x)\varphi(x') . \tag{3.10}$$

In this case one introduces a contragredient tetrad h^μ

$$h^\mu(x)=h_i^\mu t_i \tag{3.11}$$

that transforms as

$$h^\mu(x)'=\partial'_\nu x^\mu T(x)h^\nu(x')T^\dagger(x) \tag{3.12}$$

and constructs the Lagrange density L_φ from the scalar

$$I_\varphi=\frac{i}{2}\varphi^\dagger h^\mu D_\mu^\dagger\varphi+\text{H.c.} \tag{3.13}$$

The contragredient tetrad fields h_i^μ transform as

$$h_i^\mu(x)'=\partial'_\nu x^\mu V_{ij}e_j^\nu(x') , \tag{3.14}$$

where the coefficients V_{ij} are given by the traces

$$V_{ij}=tr[t_i T(x)t_j T^\dagger(x)] . \tag{3.15}$$

These coefficients are real, as are the h_i^μ's. In the next section, the metric of spacetime is made out of the two tetrads e^μ and h^μ.

Finally, if T^{-1T} is inequivalent to T, T^*, and $T^{-1\dagger}$, then there may be a field that transforms as

$$\omega(x)'=T^{-1T}(x)\omega(x') . \tag{3.16}$$

Now the appropriate scalar is

$$I_\omega=\frac{i}{2}\omega^\dagger h^{*\mu}D_\mu^T\omega+\text{H.c.} \tag{3.17}$$

The present notation does not help one remember how various fields transform. For this a notation due to van der Waerden is useful. Objects that transform with T are given a superscripted index:

$$\psi^{a\prime}=T^a{}_b\psi^b . \tag{3.18}$$

Those that transform with T^* are given a dotted superscripted index:

$$\chi^{\dot a\prime}=T^{*\dot a}{}_{\dot b}\chi^{\dot b} . \tag{3.19}$$

Those that transform with $T^{-1\dagger}$ get a dotted subscript:

$$\varphi'_{\dot a}=T^{-1\dagger}{}_{\dot a}{}^{\dot b}\varphi_{\dot b} . \tag{3.20}$$

Finally, those that transform with T^{-1T} get an undotted subscript:

$$\omega'_a=T^{-1T}{}_a{}^b\omega_b . \tag{3.21}$$

The components of the tetrad matrices are $e^\mu(x)_{\dot a b}$ and $h^\mu(x)^{a\dot b}$. Those of the curvature-tensor matrix are $F^a_{\mu\nu b}$.

Scalars now may be formed by contracting an upper undotted index with a lower undotted index or by contracting an upper dotted index with a lower dotted index. Some examples are

$$\psi^a\omega_a,\ \chi^{\dot a}\varphi_{\dot a},\ \varphi^\dagger{}_a\psi^a,\ \text{and}\ \chi^{\dagger a}\omega_a . \tag{3.22}$$

Gauge-invariant mass terms may be made from such scalars.

Other scalars may be constructed if fields like p^{ab}, $q^{\dot a b}$, r_{ab}, or $s_{\dot a b}$ are present. Some examples are

$$\omega^T p\omega,\ \omega^\dagger q\omega,\ \psi^T r\psi,\ \psi^\dagger s\psi,\ \text{and}\ \chi^T s\psi . \tag{3.23}$$

Such interactions may contribute to fermion masses if certain components of these extra fields p, q, r, or s acquire vacuum expected values. These interactions are not all of the Yukawa type since some of the components of the extra fields are vector fields. The extra fields q and s may be taken to be Hermitian and may be expanded in terms of the matrices t_i as

$$q=q_i t_i \text{ and } s=s_i t_i . \tag{3.24}$$

For later purposes it will be well to assume that q and s are nonsingular matrices.

IV. ACTIONS FOR GAUGE FIELDS

The tetrads defined in the preceding section will now be used to construct the metric of spacetime and various possible actions for the gauge fields A_μ.

One may construct two Hermitian contravariant tensors from the tetrads e^μ and h^ν. The first,

$$\begin{aligned} g^{\mu\nu} &= \tfrac{1}{2}\, tr(e^\mu h^\nu + e^\nu h^\mu) \\ &= \tfrac{1}{2}(e^\mu_{\dot a b} h^{\nu b \dot a} + e^\nu_{\dot a b} h^{\mu b \dot a}) \\ &= \tfrac{1}{2}(e^\mu_i h^\nu_i + e^\nu_i h^\mu_i)\ , \end{aligned} \tag{4.1}$$

is real and symmetric. The second,

$$\begin{aligned} k^{\mu\nu} &= \frac{i}{2}\ \mathrm{tr}(e^\mu h^\nu - e^\nu h^\mu) \\ &= \frac{i}{2}(e^\mu_{\dot a b} h^{\nu b \dot a} - e^\nu_{\dot a b} h^{\nu b \dot a}) \\ &= \frac{i}{2}(e^\mu_i h^\nu_i - e^\nu_i h^\mu_i)\ , \end{aligned} \tag{4.2}$$

is imaginary and antisymmetric. Both transform like a contravariant tensor:

$$g^{\mu\nu}(x)' = \partial'_\lambda x^\mu \partial'_\sigma x^\nu g^{\lambda\sigma}(x') \tag{4.3}$$

with a similar equation for $k^{\mu\nu}$.

The metric of spacetime will be taken to be the covariant tensor $g_{\mu\nu}$ that is the inverse of $g^{\mu\nu}$. It transforms like

$$g_{\mu\nu}(x)' = \partial_\mu x'^\lambda \partial_\nu x'^\sigma g_{\lambda\sigma}(x') \tag{4.4}$$

since

$$\partial_\mu x'^\lambda \partial'_\sigma x^\mu = \partial_\sigma x'^\mu \partial'_\mu x^\lambda = \delta^\lambda_\sigma\ . \tag{4.5}$$

In order to make a suitable Lagrange density L from a scalar I, one must multiply the scalar by a quantity that, under a gauge transformation, is multiplied by the determinant of $\partial_\mu x'^\nu$. The determinant of any second-rank covariant tensor will do, but the natural choice is

$$\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}\ . \tag{4.6}$$

There are many scalars I that may be made from the curvature tensor $F^a_{\mu\nu b}$, the two tetrads $e^\mu_{\dot a b}$ and $h^{\mu a b}$, and their Hermitian conjugates. One need only contract each contravariant index with a covariant index, each dotted superscript with a dotted subscript, and each undotted superscript with an undotted subscript. Then if the resulting scalar is not Hermitian, one may extract its Hermitian or anti-Hermitian part. It is not useful to list more than a few of the possible scalars that may so be formed.

A natural choice linear in F is

$$\begin{aligned} I_1 &= m^2 tr(e^\mu F_{\mu\nu} h^\nu) \\ &= m^2 e^\mu_{\dot a b} F^b_{\mu\nu c} h^{\mu c \dot a}\ , \end{aligned} \tag{4.7}$$

in which m is a mass, presumably the Planck mass, inserted for dimensional reasons. Two scalars quadratic in F are

$$I_2 = tr(F_{\mu\nu} F_{\lambda\sigma}) g^{\mu\lambda} g^{\nu\sigma} \tag{4.8}$$

and

$$I_3 = tr(e^\mu F_{\mu\nu} h^\nu e^\lambda F_{\lambda\sigma} h^\sigma)\ . \tag{4.9}$$

Scalars bininear in F and $F^\dagger$ tend to be lopsided like

$$I_4 = tr(F_{\mu\nu} h^\nu e^\lambda h^{\sigma\dagger} F^\dagger_{\lambda\sigma} e^{\mu\dagger})\ . \tag{4.10}$$

If, however, some of the extra fields p, q, r, or s are present, then symmetrical scalars bilinear in F and $F^\dagger$ do exist. One of the simplest is

$$I_5 = tr(F_{\mu\nu} s^{-1} F^\dagger_{\lambda\sigma} s) g^{\mu\lambda} g^{\nu\sigma}\ , \tag{4.11}$$

in which the assumed invertibility of s is used. Another is

$$I_6 = tr(e^\mu F_{\mu\nu} h^\nu s\, h^\lambda F^\dagger_{\sigma\lambda} e^\sigma s^{-1})\ . \tag{4.12}$$

There are many other possibilities.

One may convert the above scalars I_i into Lagrange densities by multiplying them by $\sqrt{-g}$. One may sometimes avoid this square root by using the totally antisymmetric Levi-Civita tensor $\epsilon^{\mu\nu\lambda\sigma}$ of which $\epsilon^{0123} = 1$. An example is

$$L_7 = \epsilon^{\mu\nu\lambda\sigma} tr(F_{\mu\nu} s^{-1} F^\dagger_{\lambda\sigma} s)\ . \tag{4.13}$$

One may wish to provide the extra field s with its own action term. A suitable covariant derivative for s is

$$D_\mu s = \partial_\mu s - s A_\mu - A^\dagger_\mu s\ , \tag{4.14}$$

which like s is Hermitian. A Lagrange density for s may be made from the scalar

$$I_s = m'^2 tr(D_\mu s\, s^{-1} D_\nu s\, s^{-1} e^\mu h^\nu)\ . \tag{4.15}$$

There are many other such scalars.

V. EXAMPLES

Four examples of collateral gauge theories are given in this section. The fourth pretends to unify gravity with the electronuclear interactions and provides natural roles for two 16-member families of quarks and leptons. These examples are intended to illustrate the structure of collateral gauge theories, not to provide realistic models.

For simplicity the gauge group G of the first example will be taken to be the direct product of SL(2,C) with a compact Lie group K that acts only on the species indices. The theory is then only trivially collateral and resembles many conventional theories. The matrices t_i may be taken to be the unit matrix and the three Pauli matrices. Since the generators of the Lorentz group commute with those of the Yang-Mills group, the curvature tensor F splits into two independent pieces. The piece referring to the Lorentz group may contribute to the action via the scalar I_1 of Eq. (4.7), in which the tetrad fields are related by $h_i^{\mu}=\eta_{ij}e_j^{\mu}$, where η is the metric of flat space. Then $k^{\mu\nu}$ vanishes and $g^{\mu\nu}=e^{\mu T}\eta e^{\nu}$. The Yang-Mills piece may contribute to the action through the scalar I_2 of Eq. (4.8).

The gauge group G of the second example is GL(4,C) and T is its fundamental representation. The vacuum expected values of the tetrads determine which gauge transformations represent Lorentz transformations, as will be explained at the end of this section. For one simple set of tetrad vacuum expected values,[3] a Lorentz transformation is represented by a matrix T of the form

$$T=\begin{bmatrix}\Lambda & 0\\ 0 & \Lambda\end{bmatrix}, \tag{5.1}$$

where Λ is in SL(2,C). There are now 16 matrices t_i which may be defined in terms of the four matrices $\sigma^i=(1,\vec{\sigma})$ as

$$t_i=\frac{1}{2}\begin{bmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{bmatrix}, \quad t_{4+i}=\frac{1}{2}\begin{bmatrix}0 & \sigma^i\\ \sigma^i & 0\end{bmatrix},$$
$$t_{8+i}=\frac{1}{2}\begin{bmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{bmatrix}, \tag{5.2}$$

and

$$t_{12+i}=\frac{1}{2}\begin{bmatrix}0 & i\sigma^i\\ -i\sigma^i & 0\end{bmatrix}.$$

In these examples, the tetrad fields all have spin 2. There is one tetrad field e_i^{μ} in the first example. In the second example, the tetrad fields e_i^{μ} split up into four spin-2 fields e_i^{μ}, e_{i+4}^{μ}, e_{i+8}^{μ}, and e_{i+12}^{μ}, for $i=1-4$. So do the h_i^{μ}. Each such spin-2 field actually contains a scalar component and two spin-1 components that transform as (1,0) and as (0,1). The extra field s, if present in the first example, is a vector field s_i. If present in the second example, s splits into four vector fields. The extra fields p and r have scalar components, but q like s is purely vectorial.

The gauge group G and representation T of the third example are those of the second, but now it is assumed that the tetrads assume a different set of vacuum expected values.[4] A Lorentz transformation now is represented by a matrix T of the form

$$T=\begin{bmatrix}\Lambda & 0\\ 0 & \Lambda^*\end{bmatrix}, \tag{5.3}$$

where Λ is in SL(2,C). The matrices t_i may be chosen as in the second example. The tetrad fields e_{ab}^{μ} now transform as

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})]^2=2[(1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0)]\oplus4(\tfrac{1}{2},\tfrac{1}{2})\oplus(\tfrac{3}{2},\tfrac{1}{2})\oplus(\tfrac{1}{2},\tfrac{3}{2}) \tag{5.4}$$

under Lorentz transformations. Thus the tetrad fields e_{ab}^{μ} and $h^{\mu ab}$, which transform similarly, each contain two spin-2 fields, four spin-1 fields, two scalars, as well as two spin-2 fields that transform as $(\frac{3}{2},\frac{1}{2})$ and as $(\frac{1}{2},\frac{3}{2})$ and two spin-1 fields that transform as (1,0) and as (0,1). If the extra field $s_{\dot{a}b}$ is present, then it transforms as

$$2(\tfrac{1}{2},\tfrac{1}{2})\oplus(1,0)\oplus(0,1)\oplus2(0,0)\ . \tag{5.5}$$

The fourth example is a toy model that seeks to unify gravity with the electronuclear interactions in a world with two families of quarks and leptons. The simplest and most conventional such model is provided by the first example with $K=SU_5$. In order to make a fully collateral theory, however, it seems necessary to avoid grand-unification schemes[5] that place fermions in tensor representations, like the $\underline{10}$ of SU_5. For such fermions would then transform as tensors under Lorentz transformations as well. The simplest scheme[6] with spinor representations[7] for fermions uses the group SO(10) and places 16-member families of fermions into $\underline{16}^+$ spinorial representations. The gauge group of the fourth example is $G=SO(1,13)$, which contains $SL(2,C)\otimes SO(10)$ as a maximal subgroup.[8] Fermions are placed in the $\underline{64}^+$ spinorial representation T of G. Under $SL(2,C) \otimes SO(10)$, the $\underline{64}^+$ transforms as $((\frac{1}{2},0),\underline{16}^+) \oplus ((0,\frac{1}{2}),\underline{16}^-)$, to which are assigned one SO(10) family and one "antifamily," i.e., the right-handed antifields of a second SO(10) family.[8] The representation $T^* = \underline{64}^-$ seems to have the wrong structure, being $((\frac{1}{2},0),\underline{16}^-) \oplus ((0,\frac{1}{2}),\underline{16}^+)$. The representations T^{-1T} and $T^{-1\dagger}$ are here equivalent to T and T^*. A Lorentz transformation is represented by a 64 by 64 matrix T that has 16 factors of the SL(2,C) matrix Λ and 16 factors of Λ^* along its main diagonal. There are now 91 matrices t_i, which may be constructed iteratively from those of the second example. One might be able to extend this toy model to one big enough to hold four families by choosing as G a somewhat larger group in which T, T^*, T^{-1T}, and $T^{-1\dagger}$ are all inequivalent.

There are many choices for the action of the gauge field. The simplest choice involves the scalar I_1. Two other choices use the scalars I_5 and I_6 of Eqs. (4.11) and (4.12).

In these examples if the scalar I_1 is used, then the field equation that follows from the requirement that the action be stationary under small variations of the gauge fields A_μ is an equation that involves A_μ only algebraically. In general this equation can be solved for A_μ in terms of the tetrad fields, the fermi fields, and any extra fields that may be present. The field equations of the tetrad fields involve derivatives of A_μ. So if one substitutes the expression for the gauge fields into the tetrad field equations, then one finds second-order differential equations for the tetrad fields.

In these examples, suitable actions for the fermi fields may be made from the scalars (3.2), (3.9), (3.14), and (3.17).

As mentioned in the Introduction, the vacuum expected values of the tetrads make the vacuum and S matrix less symmetric than the action. Whatever the action of the theory, the metric of spacetime is the inverse of the matrix $g^{\mu\nu}$, which is the trace (4.1) of the product of the tetrad fields $e^\mu_{\dot{a}b}$ and $h^{\nu b\dot{a}}$. The tetrad fields therefore assume nonzero vacuum expected values wherever the metric is well defined.. To lowest order in $\hbar$, the vacuum expected values of the tetrads are solutions of the classical equations of motion. Collateral gauge theories generally possess classical vacuum solutions in which the gauge fields vanish, $A_\mu = 0$, in which the tetrads e^μ and h^ν assume constant values, and in which the extra field s, if present, is also constant. These solutions describe empty flat space. They spontaneously break the symmetry of the vacuum under general coordinate transformations, and reduce its gauge symmetry. Some classical tetrad solutions describe curved space or have the wrong signature. Those that describe flat Minkowski space leave the vacuum with rigid Poincaré invariance, in which coordinate transformations, $x \rightarrow x'$, must be synchronized with rigid Lorentz rotations, $\psi' = T\psi$. Thus, as noted in the examples, the tetrad vacuum expected values define what is meant by a Lorentz transformation. They leave the vacuum with an internal gauge symmetry associated with a compact Lie group lying in the gauge group G. Any vacuum expected values assumed by any extra fields, such as s, may further reduce the symmetries of the vacuum and of the S matrix.

The vacuum expected values of the tetrads also specify the chiralities of the matter fields ψ^a, since they define what SL(2,C) matrices are associated with a Lorentz transformation. The index a is a pair $a = (\alpha, s)$ of indices in which the Weyl index α is 1 or 2 and the species index s runs from 1 to n. The tetrad vacuum expected values

$$e^\mu{}_{\dot{a}b} = e^\mu{}_{\dot{\alpha}s,\beta s'} = \frac{1}{\sqrt{2n}} \sigma^\mu{}_{\dot{\alpha}\beta} \delta_{s,s'} \tag{5.6}$$

and

$$h^{\nu b\dot{a}} = h^{\nu\beta s',\dot{\alpha}s} = \frac{1}{\sqrt{2n}} \sigma^{\nu\beta\dot{\alpha}} \delta_{s,s'} \tag{5.7}$$

describe flat Minkowski space and imply that the field $\psi^{\alpha s}$ consists of n left-handed Weyl spinors. These tetrad values reduce the internal symmetry of the vacuum to U(n). The symmetries of the vacuum and of the S matrix are then of the form of the direct product of the Poincaré group with a compact internal symmetry group, as required by the Coleman-Mandula theorem.[2] Their theorem suggests that something like this happens whenever it is possible to define an S matrix in flat space.

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[1] P. van Nieuwenhuizen, Phys. Rep. 68, 189 (1981).

[2] S. Coleman and J. Mandula, Phys. Rev. 159, 1251 (1967).

[3] A suitable set of tetrad vacuum expected values is $e_i^\mu=\delta_i^\mu$ and $h_i^\mu=\eta_{ij}e_j^\mu$ for $\mu,i,j=1-4$ and $e_i^\mu=h_i^\mu=0$ for $i=5-16$ and $\mu=1-4$. Space is then flat and Minkowskian and the symmetry group of the S matrix is the direct product of the Poincaré group with the internal symmetry group U(2), in accordance with the Coleman-Mandula theorem (Ref. 2).

[4] Such as $e_1^1=h_1^1=1$; $e_{8+i}^\mu=-h_{8+i}^\mu=\delta_i^\mu$ for $\mu,i=2-4$; and all others zero.

[5] H. Georgi and S. Glashow, Phys. Rev. Lett. 32, 438 (1974).

[6] H. Georgi, in *Particles and Fields—1974,* Proceedings of the 1975 Meeting of the Division of Particles and Fields of the Americal Physical Society, edited by C. Carlson (AIP, New York, 1975); H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975); and M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. 50, 721 (1978).

[7] F. Wilczek and A. Zee, Phys. Rev. 25, 553 (1982).

[8] R. Casalbuoni and R. Gatto, Phys. Lett. 90B, 81 (1980). These authors considered the group O(13,1) and noted that it naturally supports two families. They suggested gauging it, but seem not to have done so.