Regularization of the P Representation

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A representation is introduced for the density operator of the electromagnetic field that is suitable for all density operators and that reduces to the coherent-state P representation when the latter exists. It expresses the density operator ρ as the sum of four terms, each of which is a two-dimensional weighted integral over outer products of coherent states. The first integral has the form of the P representation, i.e., the outer products are projection operators. The absence of singularities in this term is achieved by the presence of the three supplementary integrals which vanish when the density operator possesses the P representation. In general, for stationary density operators, only the first two terms of the regularized P representation are necessary. A simple prescription is given for obtaining the four weight functions of this representation from the function $\langle\alpha|\rho|\alpha\rangle$, where $|\alpha\rangle$ is a coherent state and ρ is the density operator. According to this prescription, the P representation does not exist and one or more of the supplementary, regularizing terms is necessary when the function $\langle\alpha|\rho|\alpha\rangle$ contains a term that decreases more rapidly than $\exp(-|\alpha|^2)$ as $|\alpha|\to\infty$. The regularized P representation affords nonsingular integral expressions for all density operators and for most expectation values, including, when they are finite, those of the normally ordered products of the creation and annihilation operators, $a^\dagger$ and a. The construction and use of this representation is illustrated with the aid of simple examples in which the density operator does not possess the P representation.

I. INTRODUCTION

In the preceding paper,[1] we had as an objective the resolution of an ambiguity that has obscured the role of the P representation[1–9] in quantum optics. The basic question was whether the P representation exists for all electromagnetic fields or only for some limited subclass. The resolution of this ambiguity is also an objective of the present paper, in which we introduce a representation that is suitable for all electromagnetic fields and that reduces to the P representation when the latter exists.

The P representation expresses the density operator ρ for each mode of the electromagnetic field as an integral over the projection operators upon the coherent states. In the notation of I, it may be written as

$$\rho=\int|\alpha\rangle\langle\alpha|P(\alpha)d^2\alpha. \tag{1.1}$$

In the main, the effort to show that the P representation exists for all density operators has relied upon the existence of the weight function $P(\alpha)$ as a member of some space of distributions or of generalized functions.[5,7,10–17] Prior to this writing, the only space that has actually been shown to include the function $P(\alpha)$ for every density operator is the space of ultradistributions Z'.[12,17–19]

We improve upon this result in the present paper and show that $P(\alpha)$ always lies within a well-defined and relatively small subspace of Z'. More precisely, we show that the singular part of $P(\alpha)$ is at worst the Fourier transform of an infinitely differentiable function.[20]

We do not, however, draw from our much stronger result the conclusion that the P representation exists for

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[1] K. E. Cahill, preceding paper, Phys. Rev. **180**, 1239 (1969). hereafter referred to as I; equation numbers cited from it will be prefixed by I.
[2] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).
[3] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[4] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[5] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[6] R. J. Glauber, *Quantum Optics and Electronics* edited by C. de Witt *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1965), p. 63.
[7] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 234 (1965).
[8] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
[9] Although the P representation has been used principally in the description of the electromagnetic field, it applies equally well to the description of any boson field or of any system that may be characterized by a set of canonically conjugate Hermitian observables $\{q_i,p_j\}$, with $[q_i,p_j]=i\hbar\delta_{i,j}$.
[10] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
[11] J. R. Klauder, Phys. Rev. Letters **15**, 534 (1966), in the most successful departure from the approach mentioned above, has suggested for the P representation a species of universality based upon a limiting procedure. It has not been shown, however, that this procedure works for the ensemble averages of unbounded operators into which class fall a variety of physically important operators, such as the normally ordered products of the creation and annihilation operators, $(a^\dagger)^n a^m$. It should also be noted that the type of universality proved by Klauder for the P representation has been proved (and improved) in Ref. 18 for the entire spectrum of representations for the density operator that are discussed in Ref. 8. Inasmuch as many of these representations are decidedly less suitable for all density operators than even the P representation, it is clear that we are dealing here with highly attenuated types of universality.
[12] M. M. Miller and E. A. Mishkin, Phys. Rev. **164**, 1610 (1967).
[13] C. L. Mehta, Phys. Rev. Letters **18**, 752 (1967).
[14] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, 1968).
[15] G. S. Agarwal and E. Wolf, Phys. Letters **26A**, 485 (1968).
[16] G. S. Agarwal and E. Wolf, Phys. Rev. Letters **21**, 180 (1968); **21**, 656 (1968), Erratum.
[17] M. M. Miller, J. Math. Phys. **9**, 1270 (1968).
[18] K. E. Cahill, thesis, Harvard University, 1967 (unpublished; obtainable from University Microfilms, Ann Arbor), Sec. X.
[19] Although we shall not need to know anything about the space Z', which is the image of the space of distributions $\mathfrak{D}'$ under the Fourier transformation, its properties are touched upon in Appendix A. For a fuller account, see Ref. 43.
[20] We show that $P(\alpha)$ is the sum of a continuous and square-integrable function and a distribution in the space E, which is the image under the Fourier transformation of the space $\mathcal{E}$ of all infinitely differentiable functions.

all density operators.[21] Rather, by using a type of mixed Fourier-Laplace representation that has been established[22] for the Fourier transform of an infinitely differentiable function, we derive a representation that exists for all density operators and that reduces to the P representation under certain conditions. This representation may be thought of as a regularization of the P representation; we shall refer to it as the regularized P representation.

The regularized P representation does not in its general form preserve the considerable simplicity that characterizes the P representation. It has, however, a fairly elementary structure for the case of stationary density operators, i.e., those for which $[\rho, a^\dagger a]=0$. These density operators it expresses in the form

$$\rho=\int |\alpha\rangle\langle\alpha| P_1(\alpha)d^2\alpha+\int |-\alpha\rangle\langle\alpha| P_2(\alpha)d^2\alpha. \quad (1.2)$$

The first integral is of the same form as the P representation and the supplementary integral differs from it only by virtue of the minus sign in the ket $|-\alpha\rangle$.

For density operators that are not stationary and that do not possess the P representation, the regularized P representation contains two more supplementary integrals. These additional terms resemble the first two in that they are two-dimensional integrals over coherent-state dyadics but differ from them in that the arguments of the coherent states are restricted to values that are purely real in the third term and purely imaginary in the fourth. In its general form the regularized P representation may be written as

$$\rho=\int |\alpha\rangle\langle\alpha| P_1(\alpha)d^2\alpha+\int |-\alpha\rangle\langle\alpha| P_2(\alpha)d^2\alpha$$
$$+\int |x-y\rangle\langle x+y| P_3(x,y)dxdy$$
$$+\int |ix+iy\rangle\langle -ix+iy| P_4(x,y)dxdy, \quad (1.3)$$

where the integrations over the real variables x and y extend from minus infinity to plus infinity. The P representation may be said to exist only when the three supplementary integrals can be set equal to zero.

We present an elementary prescription for obtaining the weight functions P_{1-4} from the function $\langle\alpha|\rho|\alpha\rangle$. This prescription, incidentally, reveals a rather simple criterion for the existence of the P representation in terms of the asymptotic behavior of the function $\langle\alpha|\rho|\alpha\rangle$. We find that the P representation exists when the function $\langle\alpha|\rho|\alpha\rangle$ may be written as a sum of terms each of which decreases, for large values of $|\alpha|$, *less* slowly than $\exp(-|\alpha|^2)$. When, on the other hand, the function $\langle\alpha|\rho|\alpha\rangle$ contains a term that decreases more rapidly than $\exp(-|\alpha|^2)$, one or more of the supplementary integrals of Eq. (1.3) is necessary.

The weight functions of the regularized P representation are suitably nonsingular for all density operators. For the vast majority of density operators, including most of those that do not possess the P representation, they are extremely well-behaved functions.

The regularized P representation faithfully reproduces the expectation values $\mathrm{Tr}[\rho(a^\dagger)^n a^m]$ of the normally ordered products of the creation and annihilation operators.[23–26] These quantities are of primary physical interest occurring, as they do, in the expressions for the correlation functions for the electromagnetic field.[3,4] For the special case of stationary density operators, these expectation values vanish for $n\neq m$, and for $n=m$ they assume the form[27]

$$\mathrm{Tr}[\rho(a^\dagger)^n a^n]$$
$$=\int |\alpha|^{2n}[P_1(\alpha)+(-1)^n e^{-2|\alpha|^2}P_2(\alpha)]d^2\alpha. \quad (1.4)$$

The magnitude of the second term, which is zero when the P representation exists, is a measure of the accuracy of the proposition that the P representation exists for all density operators.

In Sec. II we review some relevant relations between the weight function $P(\alpha)$ of the P representation and other functions of physical interest. The mathematical basis for the regularized P representation is a theorem about the Fourier transform of an infinitely differentiable function which we explain in Sec. III. In Sec. IV we derive the regularized P representation and show how to find the four weight functions from the function $\langle\alpha|\rho|\alpha\rangle$. The properties of the regularized P representation are illustrated in Sec. V, where we construct it for some simple density operators that do not possess the P representation. The formula (1.4) and its analog for nonstationary density operators are also illustrated in these examples.

II. FOURIER TRANSFORM RELATIONS

The basic properties of the coherent states $|\alpha\rangle$ and of the displacement operators $D(\alpha)$ were set forth in Sec. II of I. In this section we use these properties to derive

[21] The problem is one of compatibility. Distributions require representations and the representations available to distributions of type E are incompatible with the single integral form of the P representation.

[22] L. Ehrenpreis, Trans. Am. Math. Soc. **101**, 52 (1961).

[23] The regularized P representation leads also, of course, to the correct values for the ensemble averages $\mathrm{Tr}(\rho F)$ of other less singular operators F. We emphasize here the normally ordered products because of their close connection with experiment and because their expectation values are known (Refs. 24–26) to lie beyond the reach of the P representation.

[24] K. E. Cahill, Phys. Rev. **138**, B1566 (1965).

[25] R. Bonifacio, L. M. Narducci, and E. Montaldi, Phys. Rev. Letters **16**, 1125 (1966).

[26] R. Bonifacio, L. M. Narducci, and E. Montaldi, Nuovo Cimento **47**, 890 (1967).

[27] The general form of Eq. (1.4), Eq. (4.19) below, contains two more integrals corresponding to the third and fourth supplementary integrals in Eq. (1.3).

some basic relations between the weight function $P(\alpha)$ of the P representation and the weight functions of other representations. These relations assume their simplest form in terms of a special type of two-dimensional Fourier transform which is well suited to functions of one or more complex variables.

If $g(\xi)$ is a function of the complex variable ξ, then we define its Fourier transform $f(\alpha)$ by the relation

$$f(\alpha)=\int \exp(\alpha\xi^*-\alpha^*\xi)g(\xi)\pi^{-1}d^2\xi. \qquad (2.1)$$

In terms of real variables, with $\alpha=x+iy$ and $\xi=u+iv$, the function $f(\alpha)$ is given by

$$f(x+iy)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[2i(yu-xv)]g(u+iv)\pi^{-1}dudv,$$

which differs from the usual two-dimensional Fourier transform only by a scale change of its arguments $\mathrm{Re}\alpha$ and $\mathrm{Im}\alpha$.

In the present notation, the processes of Fourier transformation and Fourier inversion are completely symmetrical: If the functions $f(\alpha)$ and $g(\xi)$ are related by Eq. (2.1), then the inverse relation is

$$g(\xi)=\int \exp(\xi\alpha^*-\xi^*\alpha)f(\alpha)\pi^{-1}d^2\alpha. \qquad (2.2)$$

If the functions $f_1(\alpha)$ and $f_2(\alpha)$ are the Fourier transforms of the functions $g_1(\xi)$ and $g_2(\xi)$, respectively, then Parseval's equality takes the form

$$\int f_1(\alpha)f_2(\alpha)d^2\alpha=\int g_1(\xi)g_2(-\xi)d^2\xi. \qquad (2.3)$$

As a starting point for our discussion of the P representation, we shall write the Weyl representation for an arbitrary operator F in the form[8,10,18,28–36]

$$F=\int D(-\xi)f(\xi)\pi^{-1}d^2\xi, \qquad (2.4)$$

where the displacement operator $D(\xi)$ is defined in Eq. (I2.3). In this representation the weight function $f(\xi)$ is given by the trace

$$f(\xi)=\mathrm{Tr}[FD(\xi)]. \qquad (2.5)$$

If the operator F is of the Hilbert-Schmidt type, i.e., if $\mathrm{Tr}(F^\dagger F)$ is finite, then $f(\xi)$ is a square-integrable function, and we have

$$\mathrm{Tr}(F^\dagger F)=\int |f(\xi)|^2\pi^{-1}d^2\xi. \qquad (2.6)$$

The weight function for the density operator ρ in the Weyl representation

$$\rho=\int D(-\xi)\chi(\xi)\pi^{-1}d^2\xi \qquad (2.7)$$

is the characteristic function

$$\chi(\xi)=\mathrm{Tr}[\rho D(\xi)], \qquad (2.8)$$

which is square-integrable in view of the relation

$$1\geq \mathrm{Tr}(\rho^2)=\int |\chi(\xi)|^2\pi^{-1}d^2\xi. \qquad (2.9)$$

Since the operator $D(\xi)$ is unitary, as shown by Eq. (I2.4), the modulus of the characteristic function is bounded by unity:

$$|\chi(\xi)|\leq 1. \qquad (2.10)$$

In order to derive the P representation from the Weyl representation, we first use Eqs. (2.4), (2.5), (2.7), and (2.8) to express the expectation value of an arbitrary operator F in the form

$$\mathrm{Tr}(\rho F)=\int \chi(\xi)f(-\xi)\pi^{-1}d^2\xi. \qquad (2.11)$$

Let us now observe that by writing the displacement operator $D(-\xi)$, which appears in Eq. (2.4), in its normally ordered form (I2.12) we may express the coherent state matrix element $\langle\alpha|F|\alpha\rangle$ as the integral

$$\langle\alpha|F|\alpha\rangle=\int e^{\alpha\xi^*-\alpha^*\xi-\frac{1}{2}|\xi|^2}f(\xi)\pi^{-1}d^2\xi, \qquad (2.12)$$

which we recognize as the Fourier transform of the function $\exp(-\frac{1}{2}|\xi|^2)f(\xi)$. In terms of the normally ordered characteristic function, defined by

$$\chi_N(\xi)=\mathrm{Tr}(\rho e^{\xi a^\dagger}e^{-\xi^* a})=e^{\frac{1}{2}|\xi|^2}\chi(\xi), \qquad (2.13)$$

Eq. (2.11) may be written as

$$\mathrm{Tr}(\rho F)=\int \chi_N(\xi)e^{-\frac{1}{2}|\xi|^2}f(-\xi)\pi^{-1}d^2\xi. \qquad (2.14)$$

If now the function $\chi_N(\xi)$ is square-integrable, then its Fourier transform

$$P(\alpha)=\int \exp(\alpha\xi^*-\alpha^*\xi)\chi_N(\xi)\pi^{-2}d^2\xi \qquad (2.15)$$

will also be square-integrable,[37] and by using Parseval's

[28] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publishers, Inc., New York, 1950), pp. 272–276.
[29] J. E. Moyal, Proc. Camb. Phil. Soc. **45**, 99 (1948).
[30] M. S. Bartlett and J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 545 (1949).
[31] U. Fano, Rev. Mod. Phys. **29**, 74 (1957).
[32] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **46**, 883 (1960).
[33] A. E. Glassgold and D. Holliday, Phys. Rev. **139**, A1717 (1965).
[34] J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 743 (1965).
[35] J. C. T. Pool, J. Math. Phys. **7**, 66 (1966).
[36] B. R. Mollow and R. J. Glauber, Phys. Rev. **160**, 1076 (1967); **160**, 1097 (1967).
[37] R. J. Glauber, in *Physics of Quantum Electronics*, edited by P. Kelley *et al.* (McGraw Hill Book Co., Inc., New York, 1966), p. 788.

relation (2.3), together with Eqs. (2.12) and (2.14), we may write

$$\mathrm{Tr}(\rho F)=\int P(\alpha)\langle\alpha|F|\alpha\rangle d^2\alpha. \tag{2.16}$$

This relation expresses the full content of the P representation (1.1) and is equivalent to it.

When the normally ordered characteristic function $\chi_N(\xi)$ is not square-integrable but is a tempered distribution, then a modified P representation may be said to exist with a weight function $P(\alpha)$ that is a tempered distribution. This case was discussed in Sec. III of I. In general, however, the function $\chi_N(\xi)$ is bounded only by the exponential

$$|\chi_N(\xi)|\leq e^{\frac{1}{2}|\xi|^2}, \tag{2.17}$$

and therefore does not naturally lie within the space of tempered distributions. This fact is what led us to consider the more general representation which we describe in the following sections.

III. FOURIER TRANSFORMS FOR INFINITELY DIFFERENTIABLE FUNCTIONS

We have seen that the weight function $P(\alpha)$ of the P representation is the Fourier transform of the normally ordered characteristic function $\chi_N(\xi)$. We shall show below that the function $\chi_N(\xi)$ may always be written as the sum of a square-integrable function and an infinitely differentiable function. Thus, since square-integrable functions possess square-integrable Fourier transforms, we need only find a representation for the Fourier transform of the infinitely differentiable part of $\chi_N(\xi)$ in order to regularize the P representation. We discuss such a representation in the present section.

Let us consider first, for simplicity, the representation of the Fourier transform of an infinitely differentiable function of one real variable. It has been shown by Ehrenpreis that every infinitely differentiable function $f(x)$ may be expressed as the sum of two integrals in the form[22]

$$f(x)=\int_{-\infty}^{\infty}e^{ixz}g(z)d\mu(z)+\int_{-i\infty}^{i\infty}e^{ixz}h(z)d\mu(z), \tag{3.1}$$

where $g(z)$, $h(z)$, and $\mu(z)$ are functions (not distributions) having properties that we shall presently explain. The first integral, along the real axis, has the form of a Fourier transform; the second, along the imaginary axis, has the form of a Laplace transform. When the function $f(x)$ possesses an ordinary Fourier transform, the second integral vanishes. The function $\mu(z)$ is of bounded variation[38] both on the real axis and on the imaginary axis; and, although we have written it as one function, it corresponds to different functions on the two axes. The functions $g(z)$ and $h(z)$ have the following asymptotic properties: $g(z)$ goes to zero as $|z|\to\infty$ faster than every inverse power of $|z|$, and $h(z)$ goes to zero as $|z|\to\infty$ faster than $\exp(-M|z|)$ for every value of the constant M. The two integrals in Eq. (3.1) and all of their derivatives converge uniformly for x in any bounded set. Conversely, the sum of any two such integrals represents an infinitely differentiable function.

This theorem and its analog for functions of two real variables, which we present at the end of this section, are sufficient for the derivation of the regularized P representation. If, however, we want to have some idea of the class of operators F for which the regularized P representation leads to the correct ensemble averages $\mathrm{Tr}(\rho F)$, then it will be necessary for us to consider how the Parseval equality (2.3), which holds for square-integrable functions, may be generalized to the case of infinitely differentiable functions. The generalized form of Parseval's relation, which we shall now explain, will enable us in Sec. IV to derive from Eq. (2.14) the appropriate expression in terms of the regularized P representation that replaces Eq. (2.16) for the ensemble average $\mathrm{Tr}(\rho F)$.

Let T be a distribution defined on the space $\mathcal{E}$ of all infinitely differentiable functions $f(x)$ and let $(T,f(x))$ be the number that T associates with $f(x)$. Then, for every complex number z, the function $\exp(ixz)$ is in $\mathcal{E}$ and the quantity

$$\tilde{T}(z)=(T,e^{ixz}) \tag{3.2}$$

is a function of z. We may regard $\tilde{T}(z)$ as the Fourier transform of the distribution T. In terms of this notation, the form of Parseval's relation corresponding to Eq. (3.1) must be

$$(T,f(x))=\int_{-\infty}^{\infty}\tilde{T}(z)g(z)d\mu(z)+\int_{-i\infty}^{i\infty}\tilde{T}(z)h(z)d\mu(z). \tag{3.3}$$

It has been shown by Ehrenpreis[22] that this relation holds for all $f(x)$ in $\mathcal{E}$ provided the function $\tilde{T}(z)$ is an entire exponential function of slow growth, i.e., is an entire function of z that for some constants M, n, and N is bounded by

$$|\tilde{T}(z)|\leq M(1+|z|)^n\exp(N|\mathrm{Im}z|). \tag{3.4}$$

Such functions $\tilde{T}(z)$ form a space called E'; the corresponding space of distribution T, defined on $\mathcal{E}$, is called $\mathcal{E}'$.

As a simple example of the relation (3.3), let us take T to be the nth derivative of the delta function, $T=\delta^{(n)}(x)$. Then $\tilde{T}(z)=(-iz)^n$ and Eq. (3.3) reads

$$\left.\frac{d^nf(x)}{dx^n}\right|_{x=0}=\int_{-\infty}^{\infty}(iz)^ng(z)d\mu(z)+\int_{-i\infty}^{i\infty}(iz)^nh(z)d\mu(z).$$

---

[38] A function $\mu(z)$ is of bounded variation over a region if the Riemann-Stieltjes integral $\int|d\mu(z)|$ over that region exists and is finite. In this case the differential $d\mu(z)$ is a bounded measure and may be expressed as $d\mu(z)=[f(z)+\sum_{n=0}^{\infty}c_n\delta(z-z_n)]dz$, where $f(z)$ is a piece-wise continuous and absolutely integrable function, $\delta(z)$ is the delta function, and the series $\sum_{n=0}^{\infty}|c_n|$ converges.

The space of distributions comprised of the Fourier transforms of all infinitely differentiable functions $f(x)$, in $\mathcal{E}$, is assigned the symbol E. The generalized form (3.3) of Parseval's relation affords a representation for every distribution in E in terms of two integrals involving the test functions $\tilde{T}(z)$ in E'. We note that under the Fourier transformations, $\mathcal{E}\to E$ and $\mathcal{E}'\to E'$, the roles of function and distribution are interchanged. The Fourier transform of an arbitrary infinitely differentiable function $f(x)$ in $\mathcal{E}$ must in general be interpreted as a distribution in the space E. The Fourier transform of a distribution T in $\mathcal{E}'$, on the other hand, is an entire exponential function of slow growth, a function $\tilde{T}(z)$ in E'.

We have for simplicity stated these results for the case of one real variable x and one complex Fourier transform variable z. They apply equally well to the case of several variables. The form of Eq. (3.1) which we shall need in the following section involves two real variables x and y and two complex variables z and w. In that case we have[22]

$$
\begin{aligned}
f(x,y)=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(xz+yw)}g_1(z)g_2(w)d^2\mu(z,w)\\
&+\int_{-\infty}^{\infty}\int_{-i\infty}^{i\infty} e^{i(xz+yw)}g_1(z)h_2(w)d^2\mu(z,w)\\
&+\int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty} e^{i(xz+yw)}h_1(z)g_2(w)d^2\mu(z,w)\\
&+\int_{i\infty}^{i\infty}\int_{-i\infty}^{i\infty} e^{i(xz+yw)}h_1(z)h_2(w)d^2\mu(z,w)\,, \qquad (3.5)
\end{aligned}
$$

where the functions $g_{1,2}(z)$ and $h_{1,2}(z)$ have the same properties as their counterparts in Eq. (3.1), and where $\mu(z,w)$ is a function of bounded variation on the real and imaginary axes of the z and w planes.

As we shall see in Sec. IV, the weight function $P(\alpha)$ of the P representation may always be written as the sum of a square-integrable function and a distribution in the space E. Because the singular part of $P(\alpha)$ lies in the space E, the representation afforded by Eq. (3.5) is sufficient to regularize the P representation. If the singular part of $P(\alpha)$ were not in the space E but were contained only in the space of ultradistributions Z', a representation considerably different from Eq. (3.5) would be required. The representation that would be needed in that case is outlined in Appendix A.

IV. REGULARIZED P REPRESENTATION

If the normally ordered characteristic function $\chi_N(\xi)$ were square-integrable for all density operators, the P representation would need no regularization. If χ_N were always infinitely differentiable, we could regularize the P representation by applying Eq. (3.5) as it stands. That χ_N has, in general, neither of these properties poses only a small problem, however, and, after overcoming it, we shall find it easy to construct the regularized P representation and to derive formulas for its weight functions, P_{1-4}.

In Appendix B we show that $\chi_N(\xi)$ is a continuous function of ξ. We show in Appendix C that every continuous function is the sum of a square-integrable[39] function and an infinitely differentiable function. We may, therefore, write the normally ordered characteristic function χ_N in the form

$$\chi_N(\xi)=\chi_N{}^{(\mathrm{d})}(\xi)+\chi_N{}^{(\mathrm{si})}(\xi)\,, \qquad (4.1)$$

where $\chi_N{}^{(\mathrm{d})}$ is infinitely differentiable and $\chi_N{}^{(\mathrm{si})}$ is square-integrable.

Since the term $\chi_N{}^{(\mathrm{si})}$ is square-integrable, it possesses a square-integrable[39] Fourier transform given by

$$P_{\mathrm{si}}(\alpha)=\int \exp(\alpha\xi^*-\alpha^*\xi)\chi_N{}^{(\mathrm{si})}(\xi)\pi^{-2}d^2\xi\,. \qquad (4.2)$$

The function $P_{\mathrm{si}}(\alpha)$ contributes to the weight function $P_1(\alpha)$ of the regularized P representation.

To regularize the Fourier transform of the infinitely differentiable part $\chi_N{}^{(\mathrm{d})}(\xi)$ of the normally ordered characteristic function, we use the representation (3.5). In terms of real variables, $\xi=u+iv$, with $\chi_N{}^{(\mathrm{d})}(u,v)=\chi_N{}^{(\mathrm{d})}(u+iv)$, we have

$$
\begin{aligned}
\chi_N{}^{(\mathrm{d})}(u,v)=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[2i(vx-uy)]g_1(x)g_2(y)d^2\mu(x,y)\\
&+\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\exp[2i(vz-uw)]h_1(z)h_2(w)d^2\mu(z,w)\\
&+\int_{-\infty}^{\infty}\int_{-i\infty}^{i\infty}\exp[2i(vx-uw)]g_1(x)h_2(w)d^2\mu(x,w)\\
&+\int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty}\exp[2i(vz-uy)]h_1(z)g_2(y)d^2\mu(z,y)\,, \qquad (4.3)
\end{aligned}
$$

where the functions $g_{1,2}$, $h_{1,2}$, and μ have the properties noted in Sec. III. If now, with $P_{\mathrm{si}}(x,y)=P_{\mathrm{si}}(x+iy)$, we make the identifications

$$
\begin{aligned}
&P_1(x,y)dxdy\\
&\qquad=g_1(x)g_2(y)d^2\mu(x,y)+P_{\mathrm{si}}(x,y)dxdy\,,\\
&P_2(-iw,\,iz)dzdw\\
&\qquad=-\exp[-2(z^2+w^2)]\;h_1(z)h_2(w)d^2\mu(z,w)\,,\\
&P_3(x,\,-iw)dxdw\\
&\qquad=i\;\exp(-2w^2)\;g_1(x)h_2(w)d^2\mu(x,w)\,,\\
&P_4(-iz,\,y)dzdy=i\;\exp(-2z^2)\;h_1(z)g_2(y)d^2\mu(z,y)\,, \qquad (4.4)
\end{aligned}
$$

[39] The theorem we prove in Appendix C is actually stronger than this; it implies that the function $P_{\mathrm{si}}(\alpha)$ is not only square-integrable but also continuous and that its modulus tends to zero as $|\alpha|\to\infty$.

then, on letting $z=ix$ and $w=iy$ and making a transparent change of variables in the second term of Eq. (4.3), we find

$$\chi_N(u,v)=\int P_1(x,y)\exp[2i(vx-uy)]dxdy+\int P_2(x,y)\exp[-2(x^2+y^2-ux-vy)]dxdy+\int P_3(x,y)\exp[-2(y^2-uy-ivx)]dxdy+\int P_4(x,y)\exp[-2(x^2+vx+iuy)]dxdy, \quad (4.5)$$

where the integrations over x and y extend from minus infinity to plus infinity.

The form of the regularized P representation (1.3) is now a consequence of this relation together with Eqs. (2.14)–(2.16) and the generalized form (3.3) of Parseval's relation. We may, however, sidestep some bookkeeping, which that route toward its derivation would involve, merely by noticing that the expression (4.5) for $\chi_N(u,v)$ follows from the definition (2.13) of χ_N, when the density operator ρ is given by

$$\rho=\int P_1(x,y)|x+iy\rangle\langle x+iy|dxdy+\int P_2(x,y)|-x-iy\rangle\langle x+iy|dxdy+\int P_3(x,y)|x-y\rangle\langle x+y|dxdy+\int P_4(x,y)|ix+iy\rangle\langle -ix+iy|dxdy, \quad (4.6)$$

which is Eq. (1.3). The correct form for the ensemble average of an arbitrary operator F is, therefore, from Eq. (4.6),

$$\mathrm{Tr}(\rho F)=\int P_1(\alpha)\langle\alpha|F|\alpha\rangle d^2\alpha+\int P_2(\alpha)\langle\alpha|F|-\alpha\rangle d^2\alpha+\int P_3(x,y)\langle x+y|F|x-y\rangle dxdy+\int P_4(x,y)\langle -ix+iy|F|ix+iy\rangle dxdy. \quad (4.7)$$

Our task now is to determine the class of operators F for which the function $\langle\alpha|F|\alpha\rangle$ satisfies the condition (3.4) upon Parseval's relation. When we have done this and have also taken into account the additional restriction upon $\langle\alpha|F|\alpha\rangle$ due to the part of χ_N that may not be infinitely differentiable, then we shall have established a minimum class of operators F for which Eqs. (4.6) and (4.7) are valid for all density operators ρ.

The two-variable form[22] of the condition (3.4) upon Parseval's relation is that the function $\tilde{T}(z,w)$ be an entire function of z and w that is bounded by

$$|\tilde{T}(z,w)|\le M(1+|z|+|w|)^n \times\exp[N(|\mathrm{Im}z|+|\mathrm{Im}w|)] \quad (4.8)$$

for some constants M, n, and N. In order to implement this condition, we shall have to extend the function $\langle\alpha|F|\alpha\rangle$, which corresponds to $\tilde{T}(z,w)$, from a function of two real variables, $x=\mathrm{Re}\alpha$ and $y=\mathrm{Im}\alpha$,

$$F(x,y)=\langle x+iy|F|x+iy\rangle, \quad (4.9)$$

to a function $F(z,w)$ of two complex variables z and w.

It will be helpful to define for every pair of complex numbers z and w the state

$$|z,w\rangle=e^{-\frac{1}{2}z^2-\frac{1}{2}w^2}\sum_{n=0}^{\infty}\frac{(z+iw)^n}{(n!)^{1/2}}|n\rangle. \quad (4.10)$$

When z and w are both real, the state $|z,w\rangle$ is the coherent state $|z+iw\rangle$. When z and w are complex, the state $|z,w\rangle$ and the coherent state $|z+iw\rangle$ differ only by a c number,

$$|z,w\rangle=\exp(-\tfrac{1}{2}z^2-\tfrac{1}{2}w^2+\tfrac{1}{2}|z+iw|^2)|z+iw\rangle. \quad (4.11)$$

If we denote the Hermitian adjoint of the state $|z,w\rangle$ by $\langle z,w|$, then we find

$$\langle z',w'|z,w\rangle=\exp[-\tfrac{1}{2}(z-z'^*)^2-\tfrac{1}{2}(w-w'^*)^2+i(z'^*w-zw'^*)]. \quad (4.12)$$

When $z'=z$ and $w'=w$, this becomes

$$\langle z,w|zw\rangle=\exp[|z+iw|^2-\tfrac{1}{2}(z^2+z^{*2}+w^2+w^{*2})], \quad (4.13)$$

which shows that the state $|z,w\rangle$ is not in general normalized except when z and w are both real. The inner product $\langle z^*,w^*|z,w\rangle$ is, however, unity for all z and w:

$$\langle z^*,w^*|z,w\rangle=1. \quad (4.14)$$

For an arbitrary operator F, the unique analytic extension of the function $F(x,y)$ is given by the matrix element

$$F(z,w)=\langle z^*,w^*|F|z,w\rangle. \quad (4.15)$$

When, for example, the operator F is the normally ordered product, $F=(a^\dagger)^n a^m$, then we see from Eqs. (4.11) and (4.14) that the function $F(z,w)$ is the polynomial

$$F(z,w)=(z-iw)^n(z+iw)^m. \quad (4.16)$$

Now it has been shown[8,18,40] that virtually every operator F possesses a convergent normally ordered

[40] In Refs. 8 and 18 the series (4.17) is shown to converge, in a sense which is defined there and which implies the convergence of the series (4.18), for an extremely broad class of operators F including, e.g., all whose occupation number matrix elements satisfy the inequalities $|\langle n|F|m\rangle|\le MR_1^nR_2^m(n!m!)^{\frac{1}{2}-\epsilon}$, for some M, R_1, R_2, and $\epsilon>0$ and all integers n and m.

power series expansion of the form

$$F=\sum_{n,m=0}^{\infty} f_{n,m}(a^\dagger)^n a^m, \tag{4.17}$$

where the $f_{n,m}$ are c numbers. We may, therefore, assert quite generally that the function $F(z,w)$ corresponding to a given operator F is given by the convergent[40] series

$$F(z,w)=\sum_{n,m=0}^{\infty} f_{n,m}(z-iw)^n(z+iw)^m. \tag{4.18}$$

Since this series converges, there is no difficulty in satisfying the requirement that the function $F(z,w)$ be an entire function of z and w. The condition on Parseval's relation amounts, therefore, only to the restriction that the function $F(z,w)$ satisfy the growth condition (4.8).[41] We note that this restriction is satisfied by the normally ordered products, $F=(a^\dagger)^n a^m$, as is shown explicitly by Eq. (4.16). Their expectation values, according to Eq. (4.7), are given by the integrals

$$\begin{aligned}\mathrm{Tr}[\rho(a^\dagger)^n a^m]=&\int(\alpha^*)^n(\alpha)^m P_1(\alpha)d^2\alpha\\ &+(-1)^m\int(\alpha^*)^n(\alpha)^m e^{-2|\alpha|^2}P_2(\alpha)d^2\alpha\\ &+\int(x+y)^n(x-y)^m e^{-2y^2}P_3(x,y)dxdy\\ &+(i)^{n+m}\int(x-y)^n(x+y)^m e^{-2x^2}P_4(x,y)dxdy,\end{aligned} \tag{4.19}$$

which is the general form of Eq. (1.4).

Virtually all fields of practical interest in quantum optics are characterized by the property that the ensemble averages $\mathrm{Tr}[\rho(a^\dagger)^n a^m]$ are finite for all n and m. As we show in Appendix D, this requirement implies that $\chi_N(\xi)$ is infinitely differentiable. In order to be completely general, however, we must admit that the term $P_{\mathrm{si}}(\alpha)$, which arises when χ_N is not infinitely differentiable, can lead to a singularity in the first term of Eq. (4.7) even when the analytic extension $F(z,w)$ of the function $\langle\alpha|F|\alpha\rangle$ satisfies the growth condition (4.8). To avoid this difficulty we make use of the fact that the function $P_{\mathrm{si}}(\alpha)$ is square-integrable. Thus if the function $\langle\alpha|F|\alpha\rangle$ is also a square-integrable function of α, then it and the function $P_{\mathrm{si}}(\alpha)$ satisfy the usual condition upon the standard form (2.3) of Parseval's relation. In view of Eqs. (2.6) and (2.12), this additional restriction upon $\langle\alpha|F|\alpha\rangle$ is satisfied whenever the Hilbert-Schmidt norm $[\mathrm{Tr}(F^\dagger F)]^{1/2}$ of the operator F is finite, i.e., whenever the operator F is of the Hilbert-Schmidt type.

[41] The growth condition (4.8) upon $F(z,w)$ is equivalent to the requirement that the Fourier transform of $F(z,w)$ be in $\mathcal{E}'$, i.e., that the operation $\chi_N(\xi)\rightarrow\mathrm{Tr}(\rho F)$ be linear and continuous. It is therefore a mild condition.

There are then, depending upon the nature of the normally ordered characteristic function $\chi_N(\xi)$, three classes of operators for which the trace relation (4.7) is valid. If χ_N is square-integrable, which is roughly equivalent to the existence of the P representation, then $F(x,y)$ should be square-integrable over the real xy plane. If χ_N is infinitely differentiable, then $F(z,w)$ should be entire, which is almost always true, and should satisfy the growth condition (4.8). If, and this is the general case, χ_N has neither of these properties, then $F(x,y)$ should be square-integrable and $F(z,w)$ should satisfy the growth condition (4.8). These conditions upon the operator F are certainly not necessary ones for the validity of Eq. (4.7). It would, in fact, be more realistic to regard Eq. (4.7) as a perfectly general relation, at least for ordinary purposes.[41]

For most density operators the weight functions of the general representation may be easily obtained from the function $\langle\alpha|\rho|\alpha\rangle$. If we use Eq. (4.6) to form the function $\langle\beta|\rho|\beta\rangle$, then we may write the resulting relation in the form

$$\langle\beta|\rho|\beta\rangle=\rho_1(\beta)+\rho_2(\beta)+\rho_3(\beta)+\rho_4(\beta), \tag{4.20}$$

where the functions $\rho_{1-4}(\beta)$ are given by

$$\rho_1(\beta)=\int d^2\alpha\, P_1(\alpha)e^{-|\alpha-\beta|^2}, \tag{4.21}$$

$$\rho_2(\beta)=e^{-|\beta|^2}\int d^2\alpha\, P_2(\alpha)e^{\beta\alpha^*-\beta^*\alpha-|\alpha|^2}, \tag{4.22}$$

$$\rho_3(u,v)=e^{-v^2}\int dxdy\, P_3(x,y)e^{2ivy-(u-x)^2-y^2}, \tag{4.23}$$

$$\rho_4(u,v)=e^{-u^2}\int dxdy\, P_4(x,y)e^{2iux-(v-y)^2-x^2}, \tag{4.24}$$

where $\beta=u+iv$ and $\alpha=x+iy$. These expressions are Gaussian convolutions except for the second, which is a simple Fourier transformation. Thus by using the formula

$$\left(\frac{\pi}{r}\right)^{1/2}e^{-x^2/r}=\int_{-\infty}^{\infty}dk\, e^{-2ixk-rk^2}, \tag{4.25}$$

we may secure as the inverses to Eqs. (4.21)–(4.24) the relations

$$P_1(\alpha)=\pi^{-3}\int d^2\xi\, e^{\alpha\xi^*-\alpha^*\xi+|\xi|^2}\times\int d^2\beta\, \rho_1(\beta)e^{\xi\beta^*-\xi^*\beta}, \tag{4.26}$$

$$P_2(\alpha)=\pi^{-2}e^{|\alpha|^2}\int d^2\beta\, \rho_2(\beta)e^{\alpha\beta^*-\alpha^*\beta+|\beta|^2}, \tag{4.27}$$

$$P_3(x,y)=\pi^{-5/2}e^{y^2}\int dk\, e^{2ixk+k^2}$$
$$\times\int dudv\rho_3(u,v)e^{-2i(ku+yv)+v^2}, \quad (4.28)$$

and

$$P_4(x,y)=\pi^{-5/2}e^{x^2}\int dl\, e^{2iyl+l^2}$$
$$\times\int dudv\rho_4(u,v)e^{-2i(lv+xu)+u^2}. \quad (4.29)$$

The trick, of course, is to separate properly the function $\langle\beta|\rho|\beta\rangle$ into four parts $\rho_{1-4}(\beta)$ so that the four integrals (4.26)–(4.29) converge. Usually this is a simple matter. If, for large values of $|\beta|$, the function $\langle\beta|\rho|\beta\rangle$ goes to zero slower than $\exp(-|\beta|^2)$, then the proper choice is $\rho_1(\beta)=\langle\beta|\rho|\beta\rangle$ and $\rho_2(\beta)=\rho_3(\beta)=\rho_4(\beta)=0$. In this case the P representation exists with $P(\alpha)$ given by Eq. (2.15) or (4.26). When the function $\langle\beta|\rho|\beta\rangle$ contains a part that goes to zero faster than $\exp(-|\beta|^2)$, then we must identify that part with $\rho_2(\beta)$. For if we included such a term in $\rho_1(\beta)$, then in Eq. (4.26) the integration over the variable ξ would not converge since the Fourier transform of $\rho_1(\beta)$ would decrease less slowly than $\exp(-|\xi|^2)$.

If $\langle\beta|\rho|\beta\rangle$ is asymmetric and contains a part that behaves asymptotically like

$$\exp(-bu^2-cv^2),$$

with $b<1$ and $c>1$, then this part belongs to $\rho_3(u,v)$. If, on the other hand, we have $b>1$ and $c<1$, then the choice is $\rho_4(u,v)$. The terms $\rho_3(\beta)$ and $\rho_4(\beta)$ are required only for such asymmetries in $\langle\beta|\rho|\beta\rangle$. Thus when the density operator ρ represents a stationary state of the free Hamiltonian, i.e., when $[\rho,a^\dagger a]=0$, then the function $\langle\beta|\rho|\beta\rangle$ is a function of $|\beta|$ alone and we may set $\rho_3(\beta)=\rho_4(\beta)=0$. In this case the density operator may be written as

$$\rho=\int|\alpha\rangle\langle\alpha|P_1(\alpha)d^2\alpha+\int|-\alpha\rangle\langle\alpha|P_2(\alpha)d^2\alpha, \quad (4.30)$$

which is Eq. (1.2).

We shall illustrate this method for finding the weight functions P_{1-4} with the aid of specific examples in Sec. V. Let us first, however, contrast the regularized P representation with the sort of regularization that would be required if we knew only that the functional $P(\alpha)$ were a member of the space of ultradistributions Z'. In that case, we would be able to conclude from the results of Appendix A only that all density operators possess the representation

$$\rho=\int|z,w\rangle\langle z^*,w^*|P(z,w)d^2zd^2w, \quad (4.31)$$

which is no more than a complicated way of writing the R representation[4]

$$\rho=\pi^{-2}\int|\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|d^2\alpha d^2\beta$$
$$=\pi^{-2}\int|\alpha\rangle\langle\beta|R(\alpha^*,\beta)e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}d^2\alpha d^2\beta. \quad (4.32)$$

Thus the fact that the weight functional $P(\alpha)$ may always be interpreted as a member of Z' implies not that the P representation is universal but only that the R representation is universal, which is a well-known result.[4]

V. ILLUSTRATIONS OF REGULARIZED P REPRESENTATION

In this section we illustrate the procedure for finding the weight functions P_{1-4} of the regularized P representation. In the two examples we consider, the functions P_{1-4} are simple and well behaved even though the functional $P(\alpha)$ of the P representation does not exist either as a distribution or as a tempered distribution. After constructing the regularized P representation for these families of density operators, we then use it to evaluate the ensemble averages of some physically relevant operators.

As our first example, let us consider the family of density operators

$$\rho=\frac{2}{\langle n\rangle+2}\sum_{n=0}^{\infty}\left(\frac{\langle n\rangle}{\langle n\rangle+2}\right)^n|2n\rangle\langle 2n|, \quad (5.1)$$

in which the parameter $\langle n\rangle$ represents the mean number of quanta. The density operator ρ differs from one describing a chaotic mixture,

$$\rho'=\frac{1}{\langle n\rangle+1}\sum_{n=0}^{\infty}\left(\frac{\langle n\rangle}{\langle n\rangle+1}\right)^n|n\rangle\langle n|,$$

corresponding, for example, to thermal equilibrium, in that only even numbers of quanta are present in the field described by ρ. If we set

$$c^2=\langle n\rangle/(\langle n\rangle+2), \quad (5.2)$$

then we may write

$$\langle\beta|\rho|\beta\rangle=(1-c^2)e^{-|\beta|^2}\sum_{n=0}^{\infty}\frac{(c|\beta|^2)^{2n}}{(2n)!}$$
$$=\tfrac{1}{2}(1-c^2)[e^{-(1-c)|\beta|^2}+e^{-(1+c)|\beta|^2}]. \quad (5.3)$$

Since $0<c<1$, the first term goes to zero, as $|\beta|\to\infty$, slower than $\exp(-|\beta|^2)$ while the second decreases more rapidly than $\exp(-|\beta|^2)$. Thus with

$$\rho_1(\beta)=\tfrac{1}{2}(1-c^2)e^{-(1-c)|\beta|^2},$$
$$\rho_2(\beta)=\tfrac{1}{2}(1-c^2)e^{-(1+c)|\beta|^2},$$

and $\rho_3(\beta)=\rho_4(\beta)=0$, we find by using Eqs. (4.25)–(4.27)

$$P_1(\alpha)=P_2(\alpha)=(2\pi c)^{-1}(1-c^2)\times\exp[-(c^{-1}-1)|\alpha|^2] \quad (5.4)$$

and $P_3=P_4=0$.

We may now use Eq. (1.4) or (4.19) to express the ensemble average $\mathrm{Tr}[\rho(a^\dagger)^n a^m]$ as the integral

$$\mathrm{Tr}[\rho(a^\dagger)^n a^m]=\frac{1-c^2}{2\pi c}\left[\int(\alpha^*)^n\alpha^m e^{-(c^{-1}-1)|\alpha|^2}d^2\alpha+(-1)^m\int(\alpha^*)^n\alpha^m e^{-(c^{-1}+1)|\alpha|^2}d^2\alpha\right],$$

which vanishes unless $n=m$. Thus, by putting $x=|\alpha|^2$, we find

$$\mathrm{Tr}[\rho(a^\dagger)^n a^m]=\delta_{n,m}\frac{1-c^2}{2c}\int_0^\infty x^n[e^{-(c^{-1}-1)x}+(-1)^n e^{-(c^{-1}+1)x}]dx$$
$$=\tfrac{1}{2}\delta_{n,m}n!\left(\frac{\langle n\rangle}{2}\right)^n\left(1+\frac{2}{\langle n\rangle}\right)^{n/2}\left\{\left[1+\left(\frac{\langle n\rangle}{\langle n\rangle+2}\right)^{1/2}\right]^{n+1}+(-1)^n\left[1-\left(\frac{\langle n\rangle}{\langle n\rangle+2}\right)^{1/2}\right]^{n+1}\right\}. \quad (5.5)$$

In order to introduce our second example, let us recall that the operators a and $a^\dagger$ may be expressed as

$$a=(2\hbar)^{-1/2}(\lambda q+i\lambda^{-1}p) \quad (5.6a)$$

and

$$a^\dagger=(2\hbar)^{-1/2}(\lambda q-i\lambda^{-1}p), \quad (5.6b)$$

where the operators q and p are Hermitian and canonically conjugate, $[q,p]=i\hbar$, and where λ is an arbitrary real parameter. If we alter the scale parameter λ, then the operators

$$a'=(2\hbar)^{-1/2}(\lambda' q+i\lambda'^{-1}p) \quad (5.7a)$$

and

$$a'^\dagger=(2\hbar)^{-1/2}(\lambda' q-i\lambda'^{-1}p) \quad (5.7b)$$

will have the same properties as the operators a and $a^\dagger$. In particular, the operator a' possesses a complete set of eigenstates $|\alpha\rangle'$,

$$a'|\alpha\rangle'=\alpha|\alpha\rangle',$$

corresponding to the coherent states $|\alpha\rangle$ for the operator a.

Let us consider the ground state $|0\rangle'$ in the primed system. By using Eqs. (5.6) and (5.7), we find, with $r=\lambda^{-1}\lambda'$,

$$a'^\dagger=\tfrac{1}{2}(r-r^{-1})a+\tfrac{1}{2}(r+r^{-1})a^\dagger,$$

so that we have

$${}'\langle 0|a'^\dagger|\beta\rangle=0=\tfrac{1}{2}(r-r^{-1})\beta\,{}'\langle 0|\beta\rangle+\tfrac{1}{2}(r+r^{-1})\,{}'\langle 0|a^\dagger|\beta\rangle. \quad (5.8)$$

If we define the function

$$g(\beta)=\exp(\tfrac{1}{2}|\beta|^2)\,{}'\langle 0|\beta\rangle,$$

then by using Eq. (I2.9) we may write Eq. (5.8) as the differential equation

$$\frac{dg(\beta)}{d\beta}=-\beta\left(\frac{r-r^{-1}}{r+r^{-1}}\right)g(\beta),$$

whose solution is

$$g(\beta)=g(0)\exp\left[-\frac{1}{2}\left(\frac{r-r^{-1}}{r+r^{-1}}\right)\beta^2\right].$$

Thus for the density operator $\rho=|0\rangle'\,{}'\langle 0|$ the function $\langle\beta|\rho|\beta\rangle$ is given by

$$\langle\beta|\rho|\beta\rangle=|\langle 0|0\rangle'|^2\exp\left[-\left(\frac{2}{r+r^{-1}}\right)(ru^2+r^{-1}v^2)\right], \quad (5.9)$$

where $\beta=u+iv$. By means of the normalization condition

$$1=\mathrm{Tr}\rho=\int\langle\beta|\rho|\beta\rangle\pi^{-1}d^2\beta, \quad (5.10)$$

we may secure as the value of the matrix element $|\langle 0|0\rangle'|^2$:

$$|\langle 0|0\rangle'|^2=2/|r+r^{-1}|. \quad (5.11)$$

Thus with

$$b=2r/(r+r^{-1})$$

and

$$c=2r^{-1}/(r+r^{-1}),$$

we have

$$\langle\beta|\rho|\beta\rangle=(bc)^{1/2}\exp(-bu^2-cv^2), \quad (5.12)$$

where either both $0<b\leq 1$ and $1\leq c<2$ or both $1\leq b<2$ and $0<c\leq 2$. If $b=c=1$, then the state $|0\rangle'$ is the vacuum state $|0\rangle$ and the P representation exists with $P(\alpha)=\delta^{(2)}(\alpha)$. Let us choose of the two nontrivial cases $r^2<1$ so that $0<b<1$ and $1<c<2$. Then the P representation does not exist and the proper separation of the function $\langle\beta|\rho|\beta\rangle$ is $\rho_3(\beta)=\langle\beta|\rho|\beta\rangle$ and $\rho_1=\rho_2=\rho_4=0$. By using Eqs. (4.25) and (4.28), we find

$$P_3(x,y)=\pi^{-5/2}(bc)^{1/2}e^{y^2}\int_{-\infty}^{\infty}dk\,e^{2ixk+k^2}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dudv\,e^{-2i(ku+yv)-(c-1)v^2-bu^2}$$
$$=2\pi^{-1}|r-r^{-1}|^{-1}\times\exp[-2(r^{-2}-1)^{-1}(x^2+y^2)], \quad (5.13)$$

where $r^2<1$.

For the other case, $r^2>1$, the calculation is entirely similar, with $\rho_4(\beta)=\langle\beta|\rho|\beta\rangle$ and $\rho_{1,2,3}=0$. By using Eqs.

(4.25) and (4.29) we find

$$P_4(x,y)=2\pi^{-1}|r-r^{-1}|^{-1} \times\exp[-2(r^2-1)^{-1}(x^2+y^2)], \quad (5.14)$$

and $P_{1,2,3}=0$.

For the case in which $r^2<1$, we may use Eq. (4.19) and the result (5.13) for $P_3(x,y)$ to write the ensemble averages of the normally ordered products $(a^\dagger)^n a^m$ in the form

$$\mathrm{Tr}[\rho(a^\dagger)^n a^m]=\int P_3(x,y)e^{-2y^2}(x+y)^n(x-y)^m dxdy$$
$$=\frac{2}{\pi|r-r^{-1}|}\int (x+y)^n(x-y)^m \times\exp\left[-\frac{2(r^2x^2+y^2)}{(1-r^2)}\right]dxdy. \quad (5.15)$$

If we restrict ourselves to the case $n=m$, then we may construct a generating function for these moments by forming the series

$$\sum_{n=0}^{\infty}\frac{h^n\,\mathrm{Tr}[\rho(a^\dagger)^n a^n]}{n!}$$
$$=\frac{2}{\pi|r-r^{-1}|}\int \exp\left[h(x^2-y^2)-\frac{2(r^2x^2+y^2)}{(1-r^2)}\right]dxdy$$
$$=(1-2u^2h-u^2h^2)^{1/2}, \quad (5.16)$$

where we have set $u=\frac{1}{2}|r-r^{-1}|$ and have again used Eq. (4.25). If we now compare this relation with the generating function[42] for the Legendre polynomials $P_n(z)$,

$$(1-2zt+t^2)^{-1/2}=\sum_{n=0}^{\infty}t^nP_n(z),$$

then, with the identifications $t=-iuh$ and $z=iu$, we find the result

$$\mathrm{Tr}[\rho(a^\dagger)^n a^n]=n!(-iu)^nP_n(iu), \quad (5.17)$$

where, again, $u=\frac{1}{2}|r-r^{-1}|$. This result, which we have derived for the case $r^2<1$, is the correct one for all values of r. This conclusion may be arrived at either by recognizing that both sides of Eq. (5.17) may be expressed as analytic functions of r, except for the pole at $r=0$, or by using Eqs. (4.19) and (5.14).

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[42] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea Publishing Co., New York, 1949), p. 51.

APPENDIX A

In order to view the regularized P representation in better perspective, we outline here for the space of ultradistributions Z' [43] the representation theorems[22] that correspond to those presented in Sec. III for the space E of distributions. The spaces $\mathfrak{D}$, $\mathfrak{D}'$, Z, and Z' play in this discussion the roles of the spaces $\mathcal{E}'$, $\mathcal{E}$, E', and E, respectively, in Sec. III.

The space $\mathfrak{D}$ consists of all infinitely differentiable functions $f(x)$ that are of compact support, i.e., that for some R vanish for $|x|>R$. $\mathfrak{D}'$ is the space of distributions defined on $\mathfrak{D}$. Since the normally ordered characteristic function $\chi_N(\xi)$ is continuous, as is shown in Appendix B, it is in $\mathfrak{D}'$. The space Z consists of the Fourier transforms of all functions $f(x)$ in $\mathfrak{D}$. The space of ultradistributions Z' is the space of generalized functions defined on Z. Since the Fourier transform of every distribution T in $\mathfrak{D}'$ is an ultradistribution in Z', it follows[18] that the weight function $P(\alpha)$ of the P representation, being the Fourier transform of $\chi_N(\xi)$, is in Z'.

The analog for a distribution T in $\mathfrak{D}'$ to the representation (3.1) for an infinitely differentiable function, i.e., a member of $\mathcal{E}$, is a symbolic relation of the form

$$T(x)\sim\int\int e^{ixz}j(z)d^2\mu(z), \quad (A1)$$

in which the integral over the complex z plane is two-dimensional, $j(z)$ is a suitable function, and $\mu(z)$ is a bounded measure on the complex plane. In the notation of Sec. III, the form of Parseval's relation corresponding to Eq. (3.3) is

$$(T,f(x))=\int\int \tilde{f}(z)j(z)d^2\mu(z), \quad (A2)$$

where $\tilde{f}(z)$ is the Fourier transform of the function $f(x)$. This relation holds for all $f(x)$ in $\mathfrak{D}$ in which case the entire function $\tilde{f}(z)$ is in Z.

The principal difference between the representations (3.1) and (A1) is that for the space E the integration extends over two straight lines in the complex plane while for the space Z' the integration is two-dimensional. For this reason the fact that $P(\alpha)$ is in Z' implies not the existence of the P representation, nor that of the regularized P representation, but rather only the existence of the biplanar representation (4.31) which is just the R representation (4.32).

The two forms of Parseval's relation (3.3) and (A2) differ additionally in that the space Z of functions admissible to (A2) is much smaller than the corresponding space E' for (3.3). Every $\tilde{f}(z)$ in Z is an entire function of z that is bounded, for *all* integers $n\geq 0$, by

$$|\tilde{f}(z)|\leq M_n|z|^{-n}\exp(N|\mathrm{Im}z|) \quad (A3)$$

[43] For a readable discussion of the space Z', see I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, translated by E. Saletan (Academic Press Inc., New York, 1964), Vol. I, Chaps. I and II.

for some constants M_n and $N>0$. The difference between this condition and its counterpart (3.4) has as a significant consequence that the existence of $P(\alpha)$ in Z' allows us to compute the ensemble averages of no useful operators.[18] For in place of the growth condition (4.8), which follows from (3.4), the restriction (A3) leads to the growth condition[44]

$$|\langle z^*,w^*|F|z,w\rangle| \leq M_n(1+|z|+|w|)^{-n} \times \exp[N(|\mathrm{Im}z|+|\mathrm{Im}w|)] \quad \text{(A4)}$$

for *all* integers n and some constants M_n and N. This restriction rules out not only the normally ordered products, $F=(a^\dagger)^n a^m$, but also all operators F that are not in the trace class and, within the trace class,[45] all dyadics, $F=|\psi\rangle\langle\varphi|$. The physical implications of the existence of $P(\alpha)$ as an ultradistribution, i.e., in Z', are therefore vacuous.

APPENDIX B

Our object here is to show that the normally ordered characteristic function $\chi_N(\xi)$ is a continuous function of ξ. Let us first define the anti-normally ordered characteristic function $\chi_A(\xi)$ as

$$\chi_A(\xi)=\mathrm{Tr}[\rho\exp(-\xi^*a)\exp(\xi a^\dagger)]. \quad \text{(B1)}$$

By inserting (I2.8) for the identity operator between the two exponentials in this trace, we may write $\chi_A(\xi)$ as the Fourier transform of the function $\langle\alpha|\rho|\alpha\rangle$:

$$\chi_A(\xi)=\int \exp(\xi\alpha^*-\xi^*\alpha)\,\langle\alpha|\rho|\alpha\rangle\pi^{-1}d^2\alpha. \quad \text{(B2)}$$

Now $\langle\alpha|\rho|\alpha\rangle$, being both positive and integrable, as is shown by Eq. (5.10), is an absolutely integrable function of $|\alpha|$. Thus, since the Fourier transform of every absolutely integrable function is continuous, it follows that $\chi_A(\xi)$, being the Fourier transform of $\langle\alpha|\rho|\alpha\rangle$, is continuous.[46] From Eqs. (I2.12) and (I2.13), we have further

$$\chi_N(\xi)=e^{\frac{1}{2}|\xi|^2}\chi(\xi)=e^{|\xi|^2}\chi_A(\xi), \quad \text{(B3)}$$

which shows that all three characteristic functions are continuous.[8]

APPENDIX C

Our purpose here is to show that the normally ordered characteristic function $\chi_N(\xi)$ may always be decomposed into the sum of an infinitely differentiable function $\chi_N^{(d)}(\xi)$ and a square-integrable function $\chi_N^{(si)}(\xi)$, so that for all ξ

$$\chi_N(\xi)=\chi_N^{(d)}(\xi)+\chi_N^{(si)}(\xi). \quad \text{(C1)}$$

We actually shall prove the stronger result that the function $|\chi_N^{(si)}(\xi)|^p$ is integrable not only for $p=2$ but for all positive p. To prove this we shall use two simple theorems and the fact that $\chi_N(\xi)$ is a continuous function of ξ, which was shown in Appendix B.

The two theorems have to do with infinitely differentiable functions of several real variables and sets that are closed or bounded or both, like the region $|x|\leq R$. The first theorem[47] is that for every continuous function $f(x)$ there exists a sequence of infinitely differentiable functions $g_n(x)$ that converges to $f(x)$ uniformly on any bounded region. The second theorem[48] is that if W is a closed and bounded region and U is an open region containing W, then there exists an infinitely differentiable function $h(x)$ that is equal to 1 for x in W, equal to zero for x outside of U, and bounded by 1 above and by zero below for all x.

For simplicity, we shall carry out the proof for the case of one dimension and shall state our result as the following theorem:

If $f(x)$ is a continuous function on the real line, then we may write for all x

$$f(x)=g(x)+j(x), \quad \text{(C2)}$$

where the function $g(x)$ is infinitely differentiable and where the function $j(x)$ has the property that the integral of $|j(x)|^p$ over the real line is finite for all $p>0$.

Let e_n be a sequence of numbers, all less than unity and positive, such that the series $\sum_{n=-\infty}^{\infty}(e_n)^p$ converges for all $p>0$. We might take, e.g., $e_n=\exp(-|n|-1)$. Then by the first theorem there exists a sequence of infinitely differentiable functions $g_n(x)$ such that

$$|f(x)-g_n(x)|<e_n \quad \text{for} \quad n-1\leq x\leq n+2. \quad \text{(C3)}$$

For each integer n we define M_n by the rule

$$2+|f(x)|\leq M_n \quad \text{for} \quad n-1\leq x\leq n+2. \quad \text{(C4)}$$

The numbers M_n are all finite because $f(x)$ is assumed to be continuous.

Now by the second theorem there exists a sequence of infinitely differentiable functions $h_n(x)$ such that

$$\begin{aligned} h_n(x)&=1 \quad \text{if} \quad n\leq x\leq n+1,\\ h_n(x)&=0 \quad \text{if} \quad x\leq n-\tfrac{1}{2}(M_n)^{-|n|},\\ h_n(x)&=0 \quad \text{if} \quad x\geq n+1+\tfrac{1}{2}(M_{n+1})^{-|n+1|},\\ 0\leq h_n&(x)\leq 1 \quad \text{for all} \quad x. \end{aligned} \quad \text{(C5)}$$

Then, by construction, the function $g(x)$ defined by

$$g(x)=\sum_{n=-\infty}^{\infty} h_n(x)g_n(x) \quad \text{(C6)}$$

[44] The class of operators F that satisfy the growth condition (A4) is studied in Ref. 18 where it is called $O(Z)$. Apart from $F=0$, I know of no remotely tractable examples of such operators.

[45] The trace class is defined in Ref. 26 of I.

[46] See, e.g., S. Bochner and K. Chandrasekaran, *Fourier Transforms* (Princeton University Press, Princeton, N. J., 1949), Chap. I.

[47] Reference 43, p. 141.

[48] Reference 43, p. 142.

is infinitely differentiable for all x. Moreover, the function $j(x)$ defined by

$$j(x)=f(x)-g(x) \tag{C7}$$

is easily seen to have the required integrability property. For, from Eqs. (C3)–(C5), we have for every n

$$\int_{n-1/2}^{n+1/2} |j(x)|^p dx \leq (e_n)^p+(2+M_n)^p(M_n)^{-|n|}. \tag{C8}$$

If now we sum over all integers n, then we find that the first series converges by assumption and the second by Eq. (C4). The function $|j(x)|^p$ is therefore integrable over the real line for all $p>0$,

$$\int_{-\infty}^{\infty} |j(x)|^p dx<\infty. \tag{C9}$$

This theorem is easily generalized to the complex plane and to Euclidean n space. In this way, since the function $\chi_N(\xi)$ is continuous, the validity of the decomposition (C1) is established.

APPENDIX D

For most density operators of physical interest, the expectation values of the normally ordered products $\mathrm{Tr}[\rho(a^\dagger)^n a^m]$ are all finite. In this Appendix we show that for such density operators the normally ordered characteristic function $\chi_N(\xi)$ is infinitely differentiable. It follows, therefore, that these expectation values, when finite, are given correctly by Eq. (4.19) and, further, that the requirement that the operator F be of the Hilbert-Schmidt type may be dropped in actual physical problems, leaving the growth condition (4.8) upon the function $F(z,w)$ as the sole effective restriction.[41]

Let us first observe that the derivatives of $\chi_N(\xi)$ at $\xi=0$ are related to the ensemble averages of the normally ordered products. By differentiating Eq. (2.13) we find

$$\left.\frac{\partial^{n+m}\chi_N(\xi)}{\partial\xi^n\partial(-\xi^*)^m}\right|_{\xi=0}=\mathrm{Tr}[\rho(a^\dagger)^n a^m]. \tag{D1}$$

The differentiability of χ_N at $\xi=0$ consequently depends upon whether these expectation values are finite.

In order to examine the differentiability of χ_N for $\xi\neq 0$, it will be useful to consider the antinormally ordered characteristic function $\chi_A(\xi)$, defined by Eq. (B1). Since χ_N is related to χ_A by the exponential factor $\exp(|\xi|^2)$, as shown by Eq. (B3), it is clear that χ_N is infinitely differentiable if and only if χ_A is infinitely differentiable. By using Eq. (B2), we may express the derivatives of χ_A in the form

$$\frac{\partial^{n+m}\chi_A(\xi)}{\partial\xi^n\partial(-\xi^*)^m}=\int \exp(\xi\alpha^*-\xi^*\alpha)(\alpha^*)^n\alpha^m \times\langle\alpha|\rho|\alpha\rangle\pi^{-1}d^2\alpha. \tag{D2}$$

Since $\langle\alpha|\rho|\alpha\rangle$ is non-negative and $\exp(\xi\alpha^*-\xi^*\alpha)$ is unimodular, these integrals converge for all values of ξ if they converge for $\xi=0$. Thus both χ_N and χ_A are infinitely differentiable functions when the ensemble averages of the antinormally ordered products which are given by the integrals

$$\mathrm{Tr}[\rho a^m(a^\dagger)^n]=\int(\alpha^*)^n\alpha^m\langle\alpha|\rho|\alpha\rangle\pi^{-1}d^2\alpha \tag{D3}$$

are all finite. This condition is equivalent to the requirement that the ensemble averages of the normally ordered products be finite, which is normally imposed as a ground rule in actual practice.