found that (a) for a fixed expansion velocity of the turbulent region, that the number of scattering centers required decreases as the average velocity of the scattering centers increases; and (b) for a fixed number of scattering centers, the required average velocity of the scattering centers increases for increasing expansion velocity within the turbulent region. The condition stated in (a) is rather self-evident. Statement (b) is a manifestation of the dominance of deceleration over an acceleration process. This gives rise to the importance of fluctuations. We point out that one can apply this requirement to any turbulent region in which deceleration is stronger than acceleration. It is then shown that this could be true in novae and supernovae shells.

We have shown that it is possible to obtain a power-law dependence upon the average kinetic energy for the number of particles ejected from a turbulent region. The restrictions imposed by the need for the dominance of fluctuations place restrictions upon the rate at which the volume is changing. In particular, when the total energy E is related to volume V by

$$E=\text{const}\ V^q$$

we find that

$$-\frac{4(1+a_1+a_2)}{9(a_1+a_2)}<q<-\frac{4}{9},$$

where $E_{\text{turb}}=a_1E_{\text{cr}}$ and $E_{\text{mag}}=a_2E_{\text{cr}}$. If one compares the expression for the total number of particles as a function of the average kinetic energy with the results from the fluctuation origin, it is possible to obtain further information about source requirements. We found that (a) when one considers protons, the injection momentum, P_0, is normally bounded by 0.1 GeV/$c<P_0$ $<$approximately 5 GeV/c; (b) as greater energy is put into the turbulent or magnetic mode at the expense of the other two modes, the chance for fluctuation origin to occur increases[12]; (c) if the cosmic ray mode has much more energy than the other two modes, a fluctuation origin becomes unlikely; (d) the value of P_0 for heavier particles goes approximately as Z^2.

The fluctuation mechanism is shown to be efficient enough to produce very high-energy particles within a reasonable time. A power-law injection spectrum allows the energy requirement to be relaxed.

Much of the above analysis assumes that the turbulent region is expanding. There are many occurrences in astronomical phenomena where this is true. We would like to point out that this is not the only case where the fluctuation origin of cosmic rays may apply. The main assumption is that deceleration is stronger than acceleration in the turbulent region.

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. A. Lenchek for some interesting observations and Dr. H. Ögelman for many clarifying discussions.

[12] This is true in the sense of a wider variation of γ (at a fixed $\bar{T}$) for a smaller change in P_0.

PHYSICAL REVIEW VOLUME 180, NUMBER 5 25 APRIL 1969

Pure States and the *P* Representation

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(Received 20 March 1968)

The coherent-state P representation for the density operator of the electromagnetic field is studied for the case in which the density operator represents a pure state, $\rho=|\psi\rangle\langle\psi|$. An exact and complete characterization is given of the states for which the P representation exists with a weight function $P(\alpha)$ that is a tempered distribution. These states $|\varphi\rangle$ form an exceedingly narrow class: each may be generated from a particular coherent state $|\alpha\rangle$ by the application of a finite number of creation operators, i.e., $|\varphi\rangle=[c_0+c_1a^\dagger+\cdots+c_n(a^\dagger)^n]|\alpha\rangle$, where α and the c_n are arbitrary complex numbers. For them the weight function $P(\alpha)$ is a linear combination of the two-dimensional delta function and a finite number of its derivatives. For other pure states, the function $P(\alpha)$ has singularities that are not compatible with the form of the P representation.

I. INTRODUCTION

It has been found useful in the study of quantum optics to represent the density operator ρ for the electromagnetic field in terms of the coherent states. These states $|\alpha\rangle$, which form a complete set, are the eigenstates of the photon annihilation operator a, i.e., we have $a|\alpha\rangle=\alpha|\alpha\rangle$, and their eigenvalues α cover the entire complex plane. One of the more convenient representations for the density operator is the P representation which was introduced by Glauber.[1–3]

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The P representation expresses the density operator ρ, for one mode of the radiation field, in the form

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \quad (1.1)$$

where $d^2\alpha=d(\text{Re}\alpha)d(\text{Im}\alpha)$ is a real element of area and the integration extends over the whole α plane. It affords for the expectation values of the normally ordered products $(a^\dagger)^n a^m$ the integral expressions

$$\text{Tr}[\rho(a^\dagger)^n a^m]=\int P(\alpha)(\alpha^*)^n\alpha^m d^2\alpha. \quad (1.2)$$

In terms of the P representation, the description of the electromagnetic field assumes a particularly simple form that resembles the classical description.[1–6] For this reason some attention[1–24] has been focused in the recent literature upon the question of how large a class of density operators possess the P representation. It was noted at the outset by Glauber[1–3] that for some states of the field singularities of the delta-function variety and worse would appear in the weight function $P(\alpha)$ of the P representation. Further work[8] has shown that the function $P(\alpha)$ lies outside the class of tempered distributions[25] for certain classes of density operators and that when the function $P(\alpha)$ is a tempered distribution then the matrix elements of the density operator must satisfy a set of restrictive conditions. Because of the singularities in $P(\alpha)$ for certain density operators, attempts to ascribe a species of universality to the P representation have had resort either to limiting procedures[7,9,11] or to the space of ultradistributions Z'.[16,23] We discuss some of these approaches in the paper that follows, in which we show how to represent those density operators for which $P(\alpha)$ is not a tempered distribution.

Among the larger gaps in our understanding of the P representation is the absence of any description of the class of density operators for which the function $P(\alpha)$ is a tempered distribution. As a first step toward such a description, we present in this paper an exact characterization of those pure states for which the function $P(\alpha)$ is a tempered distribution. We show that each such state $|\psi\rangle$ is associated with a particular coherent state $|\alpha\rangle$ from which it may be generated by the application of a polynomial in the creation operator $a^\dagger$. These states may be expressed in the form

$$|\psi\rangle=\left[\sum_{n=0}^{N} c_n(a^\dagger)^n\right]|\alpha\rangle, \quad (1.3)$$

where α and the c_n are arbitrary complex numbers. For them, the function $P(\alpha)$ is a linear combination of the two-dimensional delta function and a finite number of its derivatives. For pure states that are not of this form, the function $P(\alpha)$ lies outside the class of tempered distributions. Thus, for no pure state is the weight function $P(\alpha)$ a nonsingular function of the variable α.

The characterization (1.3) provides a clue to the broad outlines of the solution of the more difficult characterization problem mentioned above. Let us assume that ρ is a density operator for which the weight function $P(\alpha)$ is a tempered distribution. We may generate a family of neighboring density operators ρ' which are mixtures of ρ and an arbitrary pure state $|\varphi\rangle$ by means of the relation

$$\rho'=(1-\epsilon)\rho+\epsilon|\varphi\rangle\langle\varphi|, \quad (1.4)$$

where $0<\epsilon<1$. Now, as $\epsilon\to 0$, the density operator ρ' converges to ρ in the trace-class norm[26] and, since the P

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[25] Some elementary definitions in the theory of distributions and some of the properties of tempered distributions are reviewed in Sec. III. For a fuller account see, e.g., L. Schwartz, *Théorie des Distributions* (Hermann et Cie., Paris, 1951), Vols. I and II; L. Gårding and J. L. Lions, Nuovo Cimento Suppl. **14**, 9 (1959).

[26] A trace-class operator F is one that may be written (uniquely) in the form $F=x\rho U$ where x is a non-negative number, ρ is a density operator, and U is a unitary operator. The trace-class norm of F is given by $\|F\|=x$. See, e.g., I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions*, translated by A. Feinstein (Academic Press Inc., New York, 1964), Vol. IV, Chap. I. Now if ρ is a density operator that possesses the P representation and ρ_0 is one that does not, then $\rho'=(1-\epsilon)\rho+\epsilon\rho_0$, where $0<\epsilon<1$, is also one that does not. Also, as $\epsilon\to 0$, the sequence ρ' converges to ρ in the trace-class norm, since $\|\rho'-\rho\|=\epsilon\|\rho_0-\rho\|\leq 2\epsilon$. Thus the existence of one density operator that does not possess the P representation implies that such density operators are dense in the space of all density operators, with respect to the trace-class topology. Referring now to Eq. (1.4), we see that for ρ_0 we may choose a projection operator on virtually any state $|\varphi\rangle$, i.e., any except one that fits Eq. (1.3). To describe this situation the word "dense" is therefore totally inadequate and some stronger concept, "superdense," must be defined.

representation is linear, the weight functions $P'(\alpha)$ corresponding to the density operators ρ' are tempered distributions only for those states $|\varphi\rangle$ which satisfy the description (1.3). This statement implies much more than that the density operators for which $P(\alpha)$ is not a tempered distribution are dense in the space of all density operators with respect to the topology induced by the trace-class norm.[26]

Sections II and III are introductory in nature and contain a review of the basic properties of the coherent states and of tempered distributions. The characterization (1.3) is derived in Sec. IV.

II. COHERENT STATES AND DISPLACEMENT OPERATORS

We state in this section some of the well-known properties of the coherent states and of a set of closely related unitary operators which have been called displacement operators. The derivations of these properties are given in Ref. 3.

Let us consider a physical system whose states may be expanded in terms of an infinite but denumerable orthonormal set of basis states which we label with the non-negative integers $|n\rangle$. For the case of a single mode of the electromagnetic field the states $|n\rangle$ correspond to the states of precisely n quanta and the state $|0\rangle$ corresponds to the vacuum state. The annihilation and creation operators, a and $a^\dagger$, transform the basis states $|n\rangle$ according to the rule

$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle \tag{2.1}$$

and obey the commutation relation

$$[a,a^\dagger]=1. \tag{2.2}$$

For every complex number α the exponential

$$D(\alpha)=\exp(\alpha a^\dagger - \alpha^* a) \tag{2.3}$$

defines a unitary operator which satisfies the relation

$$D^\dagger(\alpha)=D^{-1}(\alpha)=D(-\alpha). \tag{2.4}$$

The image of the vacuum state $|0\rangle$ under the unitary transformation

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{2.5}$$

is the coherent state $|\alpha\rangle$ which is an eigenstate of the operator a with eigenvalue α,

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.6}$$

The coherent states are normalized but not orthogonal, as is shown by the formula

$$\langle\beta|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 + \beta^*\alpha) \tag{2.7}$$

for their inner product. They form a complete set of states and provide for the identity operator 1 the resolution

$$1=\int |\alpha\rangle\langle\alpha|\pi^{-1}d^2\alpha, \tag{2.8}$$

where the integration is over the whole α plane and the differential element $d^2\alpha = d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha)$ is real. The coherent states are related to the basis states $|n\rangle$ by the expansion

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}(n!)^{-1/2}\alpha^n|n\rangle. \tag{2.9}$$

The effect of the unitary operator $D(\alpha)$ upon the operators a and $a^\dagger$ is given by the relations

$$D^{-1}(\alpha)aD(\alpha)=a+\alpha, \tag{2.10a}$$

$$D^{-1}(\alpha)a^\dagger D(\alpha)=a^\dagger+\alpha^*. \tag{2.10b}$$

Because this transformation displaces the operators a and $a^\dagger$ by complex numbers, the operator $D(\alpha)$ is called a displacement operator. The displacement operators obey the multiplication law

$$D(\alpha)D(\beta)=D(\alpha+\beta)\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)] \tag{2.11}$$

in which the exponential is unimodular. The normally and antinormally ordered forms of the operator $D(\alpha)$ are given by

$$D(\alpha)=\exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^\dagger)\exp(-\alpha^* a) \tag{2.12}$$

and by

$$D(\alpha)=\exp(\tfrac{1}{2}|\alpha|^2)\exp(-\alpha^* a)\exp(\alpha a^\dagger), \tag{2.13}$$

respectively.

III. TEMPERED DISTRIBUTIONS

Distribution theory affords a precise and simple framework for the analysis of the P representation. Our purpose in this section is to review some of the basic definitions of this theory and some of the properties of tempered distributions and to state precisely a theorem which is used in Sec. IV.

Functions may be considered as rules for associating numbers with numbers. From this viewpoint, distributions are rules that associate numbers with functions. The delta function $\delta(x-2)$, for example, associates the numbers 2^n with the functions x^n.

We may interpret the P representation in these terms by writing the expectation value of an arbitrary operator F in the form

$$\mathrm{Tr}(\rho F)=\int P(\alpha)\,\mathrm{Tr}(|\alpha\rangle\langle\alpha|F)d^2\alpha$$

$$=\int P(\alpha)\langle\alpha|F|\alpha\rangle d^2\alpha. \tag{3.1}$$

Thus for each density operator ρ, the distribution $P(\alpha)$ associates the numbers $\mathrm{Tr}(\rho F)$ with the functions $\langle\alpha|F|\alpha\rangle$.

Distributions are required to associate numbers in a linear and continuous fashion with the functions on which they are defined. The functions with which distributions associate numbers are called test functions. Different classes of distributions are distinguished by the nature of their test functions and by the type of continuity which they display.

The class of tempered distributions[25] and the test functions on which they are defined are conventionally given the symbols S' and S, respectively. The class S consists of all infinitely differentiable functions that, together with their derivatives of all orders, go to zero at infinity faster than any inverse power of their argument. The class S has the useful property that under the Fourier transformation it is mapped onto itself in a one-to-one fashion. This property makes it possible to define the Fourier transform of every tempered distribution as a tempered distribution. Thus the classes S and S' each behave under the Fourier transformation like the class L_2 of square-integrable functions.

In order to add some facts to this conceptual outline, we mention that the function $\exp(-|\alpha|^2)$ is a member of S and that every member of S is a member of L_2. The delta function and all of its derivatives are tempered distributions as is every L_2 function.

The best characterization of the class S' is the following: A distribution is tempered if and only if it may be identified with a derivative of finite order of a continuous function that is bounded by a polynomial.[27] Thus if the weight function $P(\alpha)$ is a tempered distribution, then it may be expressed as

$$P(\alpha)=\partial^{2n}Q(\alpha)/\partial\alpha^n\partial(\alpha^*)^n, \tag{3.2}$$

where $Q(\alpha)$ is continuous and bounded by a polynomial in $|\alpha|$. This intuitive expression for $P(\alpha)$ is given a rigorous interpretation if we substitute it into Eq. (3.1) and integrate by parts n times with respect to α and n times with respect to α^*. The surface terms vanish, since the test function[28] $\langle\alpha|F|\alpha\rangle$ goes to zero faster than every power of $|\alpha|$, and we obtain

$$\mathrm{Tr}(\rho F)=\int Q(\alpha)\frac{\partial^{2n}\langle\alpha|F|\alpha\rangle}{\partial\alpha^n\partial(\alpha^*)^n}d^2\alpha. \tag{3.3}$$

This relation, which is a moderate distortion of the literal meaning of Eq. (3.1), is the form taken by expectation values when $P(\alpha)$ is not an ordinary function but merely a tempered distribution. The corresponding expression for the density operator is

$$\rho=\int Q(\alpha)\frac{\partial^{2n}|\alpha\rangle\langle\alpha|}{\partial\alpha^n\partial(\alpha^*)^n}d^2\alpha. \tag{3.4}$$

The result which we shall need in Sec. IV concerns the classes O'_C and O_M, both of which are contained in the class S'. The product of every member of S with every member of S' is a member of the class O'_C. The class consisting of the Fourier transforms of all members of O'_C is given the symbol O_M. The theorem we shall use later is that every member of O_M is infinitely differentiable and is, together with its derivatives of all orders, bounded by a polynomial.[29]

[27] L. Schwartz, Ref. 25, Vol. I, p. 95.

[28] The ambiguity that arises when the function $\langle\alpha|F|\alpha\rangle$ is not a test function in the space S has been discussed in Refs. 8, 12, and 14.

[29] L. Gårding and J. L. Lions, Ref. 24, Sec. 7.2.

IV. DETERMINATION OF THE STATES $|\psi\rangle$

We are now in a position to characterize exactly the rather narrow class of states $|\psi\rangle$ for which the density operator $\rho=|\psi\rangle\langle\psi|$ possesses the P representation with a weight function $P(\alpha)$ that is a tempered distribution. In the course of this analysis, it will become evident that there are no pure states for which $P(\alpha)$ is square-integrable.

Let us assume that the density operator $\rho=|\psi\rangle\langle\psi|$ corresponding to the pure state $|\psi\rangle$ possesses the P representation

$$|\psi\rangle\langle\psi|=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \tag{4.1}$$

with $P(\alpha)$ a tempered distribution. In order to determine the restrictions that this assumption places upon the state $|\psi\rangle$, we form the coherent state matrix element[18,24]

$$\langle-\beta|\psi\rangle\langle\psi|\beta\rangle=\int P(\alpha)\langle-\beta|\alpha\rangle\langle\alpha|\beta\rangle d^2\alpha$$

which, by using Eq. (2.7), we may write as

$$\langle-\beta|\psi\rangle\langle\psi|\beta\rangle=\exp(-|\beta|^2)\times\int P(\alpha)\exp(\beta\alpha^*-\beta^*\alpha-|\alpha|^2)d^2\alpha.$$

If we let $\beta=x+iy$ and $\alpha=u+iv$, then we may express the function $g(\beta)$ defined by

$$g(\beta)=\exp(|\beta|^2)\langle-\beta|\psi\rangle\langle\psi|\beta\rangle \tag{4.2}$$

in the form

$$g(x+iy)=\int P(u+iv)\exp[2i(yu+xv)-u^2-v^2]dudv.$$

The function $g(\beta)$ is therefore the Fourier transform of the function $\exp(-|\alpha|^2)P(\alpha)$; it differs from the usual two-dimensional Fourier transform only by a scale change of its arguments $\mathrm{Re}\beta$ and $\mathrm{Im}\beta$.

Now the function $\exp(-|\alpha|^2)$ is in S. Thus, if the function $P(\alpha)$ is in S', then the product $\exp(-|\alpha|^2)P(\alpha)$ is in the class O'_C and its Fourier transform $g(\beta)$ is in the class O_M. The function $g(\beta)$ must therefore be bounded by a polynomial in β if $P(\alpha)$ is in S'. If, further, $P(\alpha)$ is in L_2, i.e., is square-integrable, then $g(\beta)$ is also in L_2.

Now the modulus of the function $g(\beta)$ is equal to that of the function

$$h(\beta)\equiv\exp(|\beta|^2)\langle\psi|-\beta\rangle\langle\psi|\beta\rangle \tag{4.3}$$

which is an entire function of β, as may be seen from the expansion (2.9). Thus if $P(\alpha)$ is in S' then the entire function $h(\beta)$ is bounded by a polynomial. The function $h(\beta)$ is therefore required to be a polynomial in β if $P(\alpha)$ is in S'.

If now $P(\alpha)$ were in L_2, then $h(\beta)$ would have to be both a polynomial and an L_2 function. Thus, since no polynomial is in L_2, we have the result that $P(\alpha)$ cannot be in L_2 when the density operator ρ represents a pure state.

The condition which we wish to analyze further is that the entire function $h(\beta)$ is a polynomial if $P(\alpha)$ is in S'. This condition is satisfied whenever the state $|\psi\rangle$ is a linear combination of a finite number of the basis states $|n\rangle$. It is also true for every coherent state, i.e., for $|\psi\rangle=|\alpha\rangle$, but is not true for the superposition of two coherent states, i.e., for $|\psi\rangle=c|\alpha\rangle+d|\beta\rangle$ with $\alpha\neq\beta$.[10]

We show now that this condition requires that the state $|\psi\rangle$ be a linear combination of a finite number of the states $D(\alpha)|n\rangle$

$$|\psi\rangle=D(\alpha)\sum_{n=0}^{N}c_n|n\rangle, \tag{4.4}$$

where $D(\alpha)$ is the displacement operator defined by Eq. (2.3). In this prescription the parameter α is arbitrary but constant for each state $|\psi\rangle$ and the constants c_n are arbitrary complex numbers. An alternative description of the appropriate states $|\psi\rangle$ is that they be of the form

$$|\psi\rangle=\sum_{n=0}^{N}c'_n(a^\dagger)^n|\alpha\rangle, \tag{4.5}$$

where $|\alpha\rangle$ is a coherent state. The equivalence of the two prescriptions (4.4) and (4.5) is a simple consequence of Eqs. (2.5) and (2.10).

The condition under consideration implies that for some polynomial $h(\beta)$ we have

$$h(\beta)=f(\beta)f(-\beta), \tag{4.6}$$

where the function

$$f(\beta)\equiv\exp(\tfrac{1}{2}|\beta|^2)\langle\psi|\beta\rangle \tag{4.7}$$

is entire and bounded by

$$|f(\beta)|\leq\exp(\tfrac{1}{2}|\beta|^2). \tag{4.8}$$

Since $h(\beta)$ is a polynomial, the function $f(\beta)$ has only finitely many zeros, and we may write

$$f(\beta)=p(\beta)r(\beta), \tag{4.9}$$

where $p(\beta)$ is a polynomial containing the zeros of $f(\beta)$ and $r(\beta)$ is an entire function with no zeros.

Since the function $r(\beta)$ is entire and without zeros, it may be expressed in the form

$$r(\beta)=\exp[j(\beta)], \tag{4.10}$$

where the function $j(\beta)=\ln[r(\beta)]$ is entire. We also have from Eq. (4.8) the upper bound

$$|r(\beta)|\leq M\exp(\tfrac{1}{2}|\beta|^2) \tag{4.11}$$

for some constant M. This growth condition is sufficient, according to Hadamard's theorem[30] on entire functions of exponential type, to imply that $j(\beta)$ is a polynomial. By appealing again to the growth condition (4.11), we conclude that the polynomial $j(\beta)$ is at most of second degree,

$$j(\beta)=x\beta^2+y\beta+z. \tag{4.12}$$

If we now return to the relation (4.6), we find

$$h(\beta)=p(\beta)p(-\beta)\exp(2x\beta^2+2z) \tag{4.13}$$

which, since $h(\beta)$ is a polynomial, implies that $x=0$. Upon absorbing the constant $\exp(z)$ into the polynomial $p(\beta)$, we may write the matrix element $\langle\psi|\beta\rangle$ in the form

$$\langle\psi|\beta\rangle=p(\beta)\exp(y\beta-\tfrac{1}{2}|\beta|^2). \tag{4.14}$$

Since the coherent states form a complete set of states, the state $|\psi\rangle$ is uniquely determined by Eq. (4.14) in terms of the polynomial $p(\beta)$ and the constant y. If we write the polynomial $p(\beta)$ as

$$p(\beta)=\sum_{n=0}^{N}c_n{}^*\beta^n,$$

then we may exhibit the state $|\psi\rangle$ in the form

$$|\psi\rangle=\exp(\tfrac{1}{2}|y|^2)[\sum_{n=0}^{N}c_n(a^\dagger)^n]|y^*\rangle \tag{4.15}$$

which satisfies Eq. (4.14). This characterization of the pure states for which $P(\alpha)$ is in S' is identical with that of Eq. (4.5) and its equivalent (4.4).

If we let

$$\delta^{(2)}(\alpha)=\delta(\mathrm{Re}\alpha)\delta(\mathrm{Im}\alpha) \tag{4.16}$$

be a two-dimensional delta function, then the weight function $P(\alpha)$ for the state $|\psi\rangle$ may be expressed as a linear combination of $\delta^{(2)}(\alpha-y^*)$ and a finite number of its derivatives.

ACKNOWLEDGMENTS

I wish to thank Professor Roy J. Glauber and Professor Andrew Gleason for useful conversations.

[30] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, London, 1952), 2nd ed., p. 250.