MASS FORMULAS FOR STATIC SOLITONS *

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General explicit formulas are derived for the first quantum-mechanical correction to the mass of a static soliton in a weakly-coupled two-dimensional scalar field theory.

Microscopic processes are both quantum-mechanical and non-linear. In some of them, the quantum effects predominate over the non-linear ones. These processes may be described, if the coupling is weak, by the usual perturbation theory which treats the non-linear interaction as a perturbation of an exactly-solved, linearized quantum theory. There may well exist other physical processes, however, in which the non-linear effects are more important than the quantum ones. These should be described by an approximation that treats the quantum effects as a perturbation of an exactly-solved, non-linear classical field theory [1,2].

Let us consider the equation of motion

$$\Box \varphi(x, t) + V'[\varphi(x, t)] = 0 , \quad (1)$$

for the scalar quantum field φ where $V' = \partial V/\partial\varphi$. If φ is expressed as the sum of a classical field s and a quantum field ϵ, then this equation of motion breaks up into a non-linear one for s

$$\Box s(x, t) + V'[s(x, t)] = 0 \quad (2)$$

and a linear one for ϵ

$$\Box \epsilon(x, t) + \epsilon(x, t)\, V''[s(x, t)] = 0 \quad (3)$$

provided quadratic and higher terms in ϵ are ignored. This approximation is sensible when the theory is weakly-coupled and when the soliton or *lump* s is intense, e.g., proportional to an inverse power of the coupling constant, at least over a limited region of space.

Several methods have been advanced recently for calculating the masses of lumps [1, 2]. Some of these are systematic perturbation theories; most of them are very complicated. Our purpose here is to outline a simpler treatment of this problem and to derive general, explicitly-finite, formulas for the first quantum-mechanical correction to the mass of a static lump in a two-dimensional scalar field theory. In contrast to the methods described in ref. [2], our treatment does not involve collective coordinates or functional integrals. Moreover the contribution of the zero-frequency mode appears in our formulas as an explicit and often dominant term, whereas in the other methods [2] it is hidden in an infinite renormalization counterterm, which we avoid by the use of normal ordering.

In order to state our formulas, let us write the hamiltonian corresponding to the equation of motion (1) as

$$H = :\int \mathrm{d}x\, \tfrac{1}{2}\pi(x)^2 + \tfrac{1}{2}\varphi'(x)^2 + V[\varphi(x)] : \quad (4)$$

where the field φ and its conjugate momentum π are assumed to obey canonical equal-time commutation relations, where the prime means space derivative, and where the colons denote normal ordering with respect to the mass m of the lightest elementary massive meson of the theory. Let h_s denote the single-particle Schrödinger hamiltonian

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$$h_s = [p^2 + m^2 + U(x)]^{1/2} \quad (5)$$

where $U(x) = V''[s(x)] - m^2$, and let $h_0 = (p^2 + m^2)^{1/2}$. Then our formula for the mass of the lump (s, n_i) is

$$M(s, n_i) = E_s + \sum_i n_i \omega_i + \delta M_s \quad (6)$$

where E_s is the classical value

$$E_s = \int dx\, \tfrac{1}{2} s'(x)^2 + V[s(x)] = \int dx\, s'(x)^2 , \quad (7)$$

the n_i are non-negative integers, the ω_i are the (positive) boundstate eigenvalues of the hamiltonian h_s and the first quantum-mechanical correction δM_s is the trace

$$\delta M_s = -\tfrac{1}{4} \operatorname{tr}[(h_s - h_0)^2 / h_0] \quad (8)$$

which is finite and negative and which represents a sum over the eigenstates of h_s or h_0. If the potential $U(x)$ is reflectionless and if $(1 + |x|)|U(x)|$ is integrable, then this formula for δM_s reduces to a sum over the bound states of h

$$\delta M_s = -\frac{m}{\pi} \sum_i (\sin\theta_i - \theta_i \cos\theta_i) \quad (9)$$

which in this case lie in the range $0 \leqslant \omega_i \leqslant m$ and the angle $\theta_i = \arccos(\omega_i/m)$. The lump (s, n_i), which is an excited state of the lump $(s, 0)$, is stable if $\Sigma_i n_i \omega_i < m$.

Before deriving these formulas, let us apply them to two well-known examples. The potential of the harmonic (sine-Gordon) theory is $V(\varphi) = (m^4/\lambda) \times [1 - \cos(\sqrt{\lambda}\varphi/m)]$ and the static classical lump is $s(x) = (4m/\sqrt{\lambda}) \operatorname{arctg}(e^{mx})$. The mass of the elementary meson is m. The classical mass of the lump is $E_s = 8m^3/\lambda$. The potential $U(x)$ is reflectionless and suitably integrable, and has only one bound state at $\omega_0 = 0$. The angle θ_0 is $\pi/2$, and the formula (9) gives for the first quantum-mechanical correction the value $\delta M_s = -m/\pi$, which is the DHN result [2]. The contribution of the first term of the trace (8), corresponding to the zero-frequency mode, is $\delta M_s^{(0)} = -0.284\, m$ which is 89% of the full correction δM_s.

The potential of the kink theory is $V(\varphi) = (\lambda/4)(\varphi^2 - \mu^2/\lambda)^2$. The classical static lump is $s(x) = (\mu/\sqrt{\lambda}) \tanh(\mu x/\sqrt{2})$. The mass m of the elementary meson is $\sqrt{2}\mu$. The classical mass of the lump is $E_s = m^3/3\lambda$. The potential $U(x)$ is reflectionless and suitably integrable, and has two bound states at $\omega_0 = 0$ and $\omega_1 = \sqrt{3}\, m/2$. The angles θ_i are $\theta_0 = \pi/2$ and $\theta_1 = \pi/6$, and the formula (9) gives for the first quantum-mechanical correction the value $\delta M_s = -(3m/2\pi) \times (1 - \pi/6\sqrt{3})$, which is the DHN result [2]. There are two stable lumps in the theory at $M(s, 0) = E_s + \delta$ and $M(s, 1) = E_s + \omega_1 + \delta$. The contribution of the first term of the trace (8), corresponding to the zero-frequency mode, is $\delta M_s^{(0)} = -0.272$ m which is 82% of the full correction δM_s.

In order to derive the general formulas (6–8), let us observe that the field ϵ that satisfies the equation of motion (3) may be expressed as

$$\epsilon(x, t) = \sum_n \frac{\exp(-i\omega_n t)}{\sqrt{2\omega_n}}\, a_n f_n(x) + \text{c.c.} \quad (10)$$

where $[a_n, a_n^+] = \delta_{n,m}$ and the sum is over all the eigenstates of h_s including the continuum, the $f_n(x)$ being their wave functions. Now by substituting for φ and π in the hamiltonian (4) the expressions $\varphi = s + \epsilon$ and $\pi = \dot{\epsilon}$ and then expanding H in decreasing powers of s, we find

$$H = E_s + \sum_n \omega_n : a_n^+ a_n : , \quad (11)$$

where cubic and higher powers of ϵ have been omitted. There is a simple linear relationship between the operators a_n and a_n^+ and those, $a(k)$ and $a^+(k)$, that delete and add the elementary meson of mass m [3]. By using that relationship to respect the normal ordering of H, one may express H as

$$H = E_s + \sum_n \omega_n a_n^+ a_n + \delta M_s \quad (12)$$

where δM_s is the trace (8).

In order to derive the formula (9), we make use of 3 tricks that are valid when the potential $U(x)$ is reflectionless and when $(1 + |x|)|U(x)|$ is integrable. Trick 1 [2] is that

$$\operatorname{tr}(h_s - h_0) = \sum_i \omega_i - \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \delta(k) \frac{dk^0}{dk} . \quad (13)$$

Trick 2 [4] is that the phase shift $\delta(k)$ may be expressed as the sum over bound states

$$\delta(k) = 2 \sum_i \operatorname{arctg}[(m/k) \sin\theta_i] . \quad (14)$$

Trick 3 [4] is that

$$\int_{-\infty}^{\infty} dx\, U(x) = -4m \sum_i \sin\theta_i \,. \tag{15}$$

Now if one writes the trace (8), which is finite, as the sum of the two divergent terms

$$\delta M_s = \tfrac{1}{2}\mathrm{tr}(h_s - h_0) - \frac{1}{8\pi} \int_{-\infty}^{\infty} dx\, U(x) \int_{-\infty}^{\infty} dk\, k_0^{-1}, \tag{16}$$

then, after using the 3 tricks and integrating by parts, without dropping the surface terms, one arrives at eq. (9).

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