Scalar–vector instantons in *n* dimensions: Surface terms

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By an analysis that respects surface terms, it is shown that the equations of motion for a non-Abelian gauge field A_μ^a coupled to a scalar field Φ^i possess no regular, finite-action solutions in n-dimensional Euclidean space except for $n=4$, with Φ trivial, and for $n<4$.

It has been known for about a year that the equations of motion for a non-Abelian gauge field A_μ^a coupled to a scalar field Φ^i have no regular[1], finite-action solutions in n-dimensional Euclidean space except for $n=4$, with Φ trivial, and for $n<4$. Most proofs[2] of this result seem, however, to assume the absence of certain surface terms. The purpose of the present note is to provide a demonstration that is independent of that assumption.

The action functional S will be taken as the integral $\int d^n x\, \mathcal{L}(x)$ of the Lagrange density

$$\mathcal{L}(x)=\tfrac{1}{4}F_{\mu\nu}^a(x)^2+\tfrac{1}{2}\left|D_\mu^i\Phi(x)\right|^2+V[\Phi^i(x)],$$

where

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a-ef_{abc}A_\mu^b A_\nu^c$$

and

$$D_\mu^i\Phi=\partial_\mu\Phi^i+ieA_\mu^a t_{ij}^a\Phi^j.$$

The matrices t^a are the hermitian generators of a representation g of a compact Lie group G whose structure constants are f_{abc}. The potential V is assumed to be nonnegative.

Under the Derrick[3] transformation, $\Phi(x,\lambda)=\Phi(\lambda x)$ and $A_\mu(x,\lambda)=\lambda A_\mu(\lambda x)$, the action becomes

$$S(\lambda)=\lambda^{4-n}S_A+\lambda^{2-n}S_\Phi+\lambda^{-n}S_V,$$

where S_A, S_Φ, and S_V are the contributions of the vector field A_μ, the scalar field Φ, and the potential V evaluated at $\lambda=1$. These terms are all positive and finite if the fields A_μ and Φ are nontrivial and of finite action. The derivative of $S(\lambda)$ at $\lambda=1$

$$S'(1)=(4-n)S_A+(2-n)S_\Phi-nS_V$$

is strictly negative for $n=4$, unless Φ is trivial, and for $n>4$.

It will now be shown that this same derivative $S'(1)$ is positive or zero if the fields $A_\mu(x,1)=A_\mu(x)$ and $\Phi(x,1)=\Phi(x)$ constitute a twice continuously differentiable, finite-action solution of the field equations. The demonstration uses an interesting property of the radial gauge, $x_\mu A_\mu^a(x)=0$, whose universality may be proved by an obvious modification of a well-known argument.[4] The derivatives of the fields $A_\mu(x,\lambda)$ and $\Phi(x,\lambda)$ with respect to λ at $\lambda=1$ are

$$\Phi'^i(x,1)=x_\nu\partial_\nu\Phi^i(x)$$

and

$$A_\mu'^a(x,1)=A_\mu^a(x)+x_\nu\partial_\nu A_\mu^a(x).$$

In the radial gauge, these derivatives are

$$\Phi'^i(x,1)=x_\nu D_\nu^i\Phi(x)$$

and

$$A_\mu'^a(x,1)=x_\nu F_{\nu\mu}^a(x).$$

Let $S(\lambda,<R)$ denote the action due to the fields $A_\mu(x,\lambda)$ and $\Phi(x,\lambda)$ for $|x|\le R$. Then from the assumption that the fields $A_\mu(x,1)$ and $\Phi(x,1)$ constitute a twice continuously differentiable solution of the field equations and from the compactness of the region $|x|\le R$, it follows that the derivative of $S(\lambda,<R)$ with respect to λ at $\lambda=1$ is the integral

$$S'(1,<R)=R^{n-2}\int d\Omega\; x_\rho[F_{\rho\mu}^a A_\mu'^a+\tfrac{1}{2}(D_\rho^i\Phi)^*\Phi'^i+\tfrac{1}{2}(\Phi'^i)^*D_\rho^i\Phi]$$

over the surface of the sphere $|x|=R$. In the radial gauge, this surface term is

$$S'(1,<R)=R^{n-2}\int d\Omega\;(x_\rho F_{\rho\mu}^a)^2+|x_\rho D_\rho^i\Phi|^2\ge 0,$$

which cannot be negative.

The action $S(\lambda,>R)$ due to the fields $A_\mu(x,\lambda)$ and $\Phi(x,\lambda)$ for $|x|>R$ consists of three terms that may be written as

$$\begin{aligned}S(\lambda,>R)&=S_A(\lambda,>R)+S_\Phi(\lambda,>R)+S_V(\lambda,>R)\\&=\lambda^{4-n}S_A(1,>\lambda R)+\lambda^{2-n}S_\Phi(1,>\lambda R)\\&\quad+\lambda^{-n}S_V(1,>\lambda R)\end{aligned}$$

in an obvious notation. Thus

$$\begin{aligned}S'(1,>R)&=(4-n)S_A(1,>R)+(2-n)S_\Phi(1,>R)\\&\quad-nS_V(1,>R)+R\,\partial S(1,>R)/\partial R.\end{aligned}$$

If the action of the hypothetical solution $A_\mu(x,1)$ and $\Phi(x,1)$ is finite, then the first three terms on the right-hand side of this equation can be made arbitrarily small by increasing R sufficiently. Similarly, there must be a sequence of points R_i, tending to infinity, on which the product $R_i\;\partial S(1,>R_i)/\partial R$ tends to zero; for otherwise the action would be at least logarithmically divergent. Thus, since

$$S'(1)=S'(1,<R_i)+S'(1,>R_i)\ge S'(1,>R_i),$$

it follows that $S'(1)\ge 0$ as promised.

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Suppose now that the potential $V(\Phi^a)$ is invariant under the action of the group G, that $\Phi_0 \neq 0$ is one of its minima, and that the homotopy group $\pi_{n-1}(G/H) \neq 0$, where H is the little group of Φ_0. One may look for the field configuration that has the least action in each homotopy class. The fields

$$\Phi(x) = f(|x|)g(\hat{x})\Phi_0$$

and

$$t^a A^a_\mu(x) = (i/e)h(|x|)\partial_\mu g(\hat{x})g^{-1}(\hat{x}),$$

where $\hat{x} = x/|x|$ will have finite action if f and h are smooth functions that rise (h like x^2) from zero at $x=0$ to one at $|x|=r$ and that remain equal to one for $|x|>r$. A very crude estimate of the action of such a configuration is

$$S_A \approx e^{-2}r^{n-4}, \quad S_\Phi \approx r^{n-2}, \quad \text{and} \quad S_V \approx r^n.$$

Thus, for $n>4$, the singular configurations with $r=0$ all have minimal action, $S=0$. Also for $n=4$, and Φ nontrivial, the singular configurations with $r=0$ have minimal action S equal to that of the same system without the scalar field, provided the latter possesses suitable solutions.[5] For $n<4$, the least action is that of the solution to the field equations, whenever the latter exists.

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[1]By regular is meant twice continuously differentiable. The solutions of elliptic partial-differential equations are typically analytic, so this assumption is not unduly restrictive. See, e.g., C. Miranda, *Partial Differential Equations of Elliptic Type* (Springer-Verlag, Berlin, 1970), pp. 214–15.

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