Topological cohesion

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It is shown that suitably regular, finite-energy solutions of the Yang–Mills–Higgs equations are nondissipative whenever their initial data are topologically significant.

It is well known that under certain conditions topologically stable field configurations can arise as regular solutions of the classical field equations that describe the locally gauge-invariant interaction of scalar Higgs mesons with vector Yang–Mills mesons.[1] The purpose of the present paper is to show that under very general conditions every suitably regular, finite-energy solution of the field equations for such a system is nondissipative provided that its initial data are topologically significant. This effect, which we have called topological cohesion, means that topological solitons commonly occur in such theories. It also suggests that they may exhibit some of the resiliency associated with the stricter meaning of the term soliton.[2]

The interaction of the scalar mesons ϕ^i with the vector mesons A^a_μ will be assumed to be described by a Lagrange density L that is invariant, in the usual Yang–Mills way, under the local action of a compact Lie group G. We shall not need the explicit form of L in what follows. It will be sufficient to know that the potential or self-interaction $V(\phi)$ of the Higgs mesons is a continuous, nonnegative, G-invariant function of the ϕ^i and that the points ϕ on which $V(\phi)$ assumes its minimum value, zero, form a smooth compact manifold M. It will also be assumed that the Hamilton density H derived from L is bounded below by V, as is usually the case, and that $V(\phi)$ itself is bounded below by some fixed positive number for sufficiently large $\|\phi\|$.

Since the vacuum manifold M is smooth, it follows[3] that there exists a (G-invariant) tubular neighborhood $N(M)\supset M$ and a continuous map r from $N(M)$ into M that is the identity on M itself, i.e., for ϕ in M, $r(\phi)=\phi$. Because $N(M)$ is open, its complement is closed; and so on it the continuous function $V(\phi)$, which avoids zero for large $\|\phi\|$, is bounded below by some fixed positive number ϵ. Suppose now that $\phi^i(x,t)$ and $A^a_\mu(x,t)$ form a solution of the field equations with finite energy $E=\int d^nxH(x,t)$. Then since $H\geq V\geq\epsilon$ on the complement of $N(M)$, the fields $\phi^i(x,t)$ must for all times t lie inside $N(M)$ for all points x that lie outside a region $P(t)$ whose volume $v(P(t))$ is never larger than E/ϵ. If the region $P(t)$ does not have unbounded horns or whiskers, then it can be put inside a sphere of radius $R(t)$ centered about the origin. In this paper we use the term *suitably regular* to denote a finite-energy solution $\phi^i(x,t)$ and $A^a_\mu(x,t)$ that is continuous as a function of x and t and for which the radius $R(t)$ is bounded on compact time intervals. It is likely[4] that reasonable initial data, $\phi^i(x,0)$, $A^a_\mu(x,0)$ and $\dot\phi^i(x,0)$, $\dot A^a_\mu(x,0)$, lead to solutions that are suitably regular for positive t, at least for potentials $V(\phi)$ that are physically admissible.

If now $\phi^i(x,t)$ is suitably regular, then the point $\phi(\rho\hat{x},t)$, where $\hat{x}=x/\|x\|$, lies in the neighborhood $N(M)$ for all $\rho\geq R(t)$ and $t\geq 0$. The function $r(\phi(\rho\hat{x},t))$ is then, for such fixed t and ρ, a continuous map from the sphere S^{n-1} into M and therefore falls into one of the homotopy classes of $[S^{n-1},M]$. The function $r(\phi(\rho\hat{x},t))$, for fixed t, is also a continuous map from $S^{n-1}\times[R(t),\infty)$ into M. Thus it is a homotopy between the maps $r(\phi(\rho\hat{x},t))$ and $r(\phi(\rho'\hat{x},t))$ for all pairs ρ, $\rho'\geq R(t)$. The homotopy class of ϕ therefore is independent of ρ. It is also independent of t, since for fixed ρ, greater than the upper bound of $R(t)$ on the compact interval $[0,t]$, the function $r(\phi(\rho\hat{x},t))$ is a continuous map from $S^{n-1}\times[0,t]$ into M, which makes it a homotopy between $r(\phi(\rho\hat{x},0))$ and $r(\phi(\rho\hat{x},t))$. In this paper a suitably regular, finite-energy solution is said to possess *topologically significant* initial data if this homotopy class $[\phi]$ is nontrivial. For such solutions, the name of the class $[\phi]$ may be viewed as a conserved topological charge.

Suppose now that $\phi(x,t)$, $A_\mu(x,t)$ is a suitably regular, finite-energy solution with topologically significant initial data. Then for all $t\geq 0$, there is some point $x(t)$ for which $\phi(x(t),t)$ lies outside the tubular neighborhood $N(M)$. For if at some time $t\geq 0$, $\phi(x(t),t)$ were in $N(M)$ for all x in R^n, then the function $r(\phi(\rho\hat{x},t))$ for that value of t would be a continuous map from $S^{n-1}\times[0,\infty)$ into M. It would then be a homotopy between $r(\phi(0,t))$ which is in the trivial class and $r(\phi(\rho\hat{x},t))$ which or $\rho\geq R(t)$ is in $[\phi]$ assumed nontrivial. This contradiction means that for all $t\geq 0$ there is some point $x(t)$ for which $\phi(x(t),t)$ lies outside $N(M)$. But at that point $x(t)$ the energy density $H(x(t),t)\geq V(\phi(x(t),t))\geq\epsilon$ since V is bounded below by ϵ on the complement of $N(M)$. Every suitably regular, finite-energy solution with topologically significant initial data is therefore nondissipative.

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[1]S. Coleman, in *New Phenomena in Subnuclear Physics* (Erice 1975, Part A, ed. Zichichi) (Plenum, New York, 1977).

[2]A. Scott, F. Chu, and D. McLaughlin, Proc. IEEE **61**, 1443 (1973).

[3]G. Bredon, *Introduction to Compact Transformation Groups* (Academic, New York, 1972), p. 85, Lemma 5.1, Theorem 5.4.

[4]C. Parenti, F. Strocchi, and G. Velo, Nuovo Cimento B **39**, 147 (1977).

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