Nonlinear internal symmetry

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A theory of n scalar fields is outlined in which the equations of motion are invariant under all nonsingular global transformations of the fields amongst themselves, whether linear or nonlinear.

It was recently shown to be possible to formulate a gauge theory of a noncompact internal symmetry group.[1,2] This formalism will be used in the present paper to construct a theory that posseses a high degree of nonlinear internal symmetry. A theory of n scalar fields $\phi_i(x)$ will be exhibited whose Lagrangian is invariant under all nonsingular transformations of the form

$$\phi'_i(x) = F_i[\phi_1(x), \phi_2(x), \ldots, \phi_n(x)]. \tag{1}$$

These transformations may be arbitrarily nonlinear as long as they are one-to-one and continuously differentiable. Implementation of this symmetry excludes the possibility of an explicit mass term for the fields $\phi_i(x)$. A particular solution of the field equations will be presented.

The symmetry transformation (1) is global since the functions F_i do not depend explicitly upon the space-time point x. Therefore the derivatives of the fields ϕ_i transform linearly

$$\phi'_{i,\mu}(x) = J_{ij}(x)\phi_{j,\mu}(x) \tag{2}$$

where $J_{ij}(x)$ is the partial derivative of the function $F_i[\phi(x)]$ with respect to $\phi_j(x)$

$$J_{ij}(x) = \partial F_i[\phi(x)]/\partial\phi_j(x), \tag{3}$$

and the subscript comma mu means $\partial/\partial x^\mu$. The derivatives $\phi_{i,\mu}$ transform as a vector under the group GL(n, $\mathbb{C}$). In order to make objects that are invariant under this transformation, it is necessary to introduce a metric tensor $g_{ij}(x)$ that is Hermitian and positive and that transforms as

$$g_{ij}(x) = [J^{-1}(x)]^*{}_{ki} g_{kl}(x)[J^{-1}(x)]_{lj}\,. \tag{4}$$

In matrix notation, Eqs. (2) and (4) read

$$\phi'_{,\mu}(x) = J(x)\,\phi_{,\mu}(x) \tag{5}$$

and

$$g'(x) = J^{-1\dagger}(x)\,g(x)\,J^{-1}(x)\,. \tag{6}$$

Evidently the form

$$L(\phi) = \phi^\dagger_{,\mu}(x)\,g(x)\,\phi^{,\mu}(x) \tag{7}$$

is invariant.

A suitable covariant derivative of the metric tensor g may be defined as

$$g(x)_{;\mu} = g(x)_{,\mu} + g(x)A_\mu(x) + A^\dagger_\mu(x)\,g(x) \tag{8}$$

provided the $n\times n$ matrix of gauge fields A_μ transforms as

$$A'_\mu(x) = J(x)A_\mu(x)J^{-1}(x) + J(x)_{,\mu}J^{-1}(x)\,. \tag{9}$$

For then it follows from equations (6), (8), and (9) that $g_{;\mu}$ transforms like g

$$g(x)'_{;\mu} = J^{-1\dagger}(x)\,g(x)_{;\mu}J^{-1}(x)\,. \tag{10}$$

The curvature tensor is

$$F_{\mu\nu}(x) = A_\mu(x)_{,\nu} - A_\nu(x)_{,\mu} + [A_\mu(x), A_\nu(x)] \tag{11}$$

and transforms as

$$F'_{\mu\nu}(x) = J(x)\,F_{\mu\nu}(x)\,J^{-1}(x)\,. \tag{12}$$

The Lagrange density

$$L = \phi^\dagger_{,\mu}\,g\phi^{,\mu} + \tfrac{1}{4}m^2\mathrm{tr}(g_{;\mu}\,g^{-1}g^{;\mu}g^{-1}) - \tfrac{1}{2}e^{-2}\,\mathrm{tr}(F^\dagger_{\mu\nu}\,gF^{\mu\nu}g^{-1}) \tag{13}$$

is invariant under the nonlinear internal symmetry transformation defined by equations (1), (4), and (9). The number e is a dimensionless coupling constant and m is mass.

The covariant derivatives of $F^{\sigma\tau}$ and $g^{;\nu}$ are

$$F^{\sigma\tau}_{;\mu} = F^{\sigma\tau}_{,\mu} + [F^{\sigma\tau}, A_\mu] \tag{14}$$

and

$$g^{;\nu}_{;\mu} = g^{;\nu}_{,\mu} + g^{;\nu}A_\mu + A^\dagger_\mu\,g^{;\nu}. \tag{15}$$

In terms of them, the variational equations of the space-time integral of L are:

$$\phi^{;\mu}_{;\mu} + g^{-1}g_{,\mu}\,\phi^{,\mu} = 0 \tag{16}$$

$$F^{\mu\nu}_{;\nu} = [F^{\mu\nu}, g^{-1}g_{;\nu}] - \tfrac{1}{2}e^2m^2g^{-1}g^{;\mu} \tag{17}$$

$$g^{;\mu}_{;\mu} = g_{;\mu}\,g^{-1}g^{;\mu} + e^{-2}m^{-2}g[g^{-1}F^\dagger_{\mu\nu}\,g, F^{\mu\nu}] + 2m^{-2}\,g\phi^{,\mu}\phi^\dagger_{,\mu}\,g\,. \tag{18}$$

The $\mu = 0$ component of the equation for $F^{\mu\nu}$ is a constraint, which may be called Gauss's law. By using it, one may identify the Hamiltonian density as

$$H = \phi^\dagger_{,\mu}g\phi_{,\mu} + \tfrac{1}{4}m^2\,\mathrm{tr}[(g_{;\mu}\,g^{-1})^2] + \tfrac{1}{2}e^{-2}\mathrm{tr}(F^\dagger_{\mu\nu}\,gF_{\mu\nu}\,g^{-1}) \tag{19}$$

in which each term is positive as long as g is positive.

The antisymmetry of $F^{\mu\nu}$ implies that the $n\times n$ matrix of currents

$$j^\mu = [F^{\mu\nu}, g^{-1}g_{;\nu} - A_\nu] - \tfrac{1}{2}e^2m^2\,g^{-1}g^{;\mu} \tag{20}$$

is conserved, $j^\mu_{,\mu} = 0$. By combining this conservation law with the field equation for g, one may derive the constraint

$$\phi^*_{i,\mu}\,\phi^{;\mu}_j = 0 \tag{21}$$

for all i, j, and x.

A special class of simple solutions of the field equations (16)–(18) is given by the choice $g = 1$, $A_\mu = 0$, and $\phi_i(x) = f_i(k_i\cdot x)$ where the f's are arbitrary functions and the k's form a set of colinear null vectors, $k_i\cdot k_j = 0$. None of these solutions has finite energy, except the choice $k_i = 0$ for all i. The superposition of two such special solutions is not in general a solution since the constraint (21) is nonlinear.

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[1]K. Cahill, Phys. Rev. D **18**, 2930 (1978) and D **20**, 2636 (1979).

[2]The phenomenological implications of a GL(5, ℂ) gauge theory have been discussed by Jihn E. Kim and A. Zee in the University of Pennsylvania preprints UPR–0131T and –0132T (1979).

[3]In equation (19) all the Lorentz indices are down.