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Spinors of spin-one-half fields

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Abstract

This paper reviews how a two-state, spin-one-half system transforms under rotations. It then uses that knowledge to explain how momentum-zero, spin-one-half annihilation and creation operators transform under rotations. The paper then explains how a spin-one-half field transforms under rotations. The momentum-zero spinors are found from the way spin-one-half systems transform under rotations and from the Dirac equation. Once the momentum-zero spinors are known, the Dirac equation immediately yields the spinors at finite momentum. The paper then shows that with these spinors, a Dirac field transforms appropriately under charge conjugation, parity, and time reversal. The paper also describes how a Dirac field may be decomposed either into two 4-component Majorana fields or into a 2-component left-handed field and a 2-component right-handed field. Wigner rotations and Weinberg's derivation of the properties of spinors are also discussed.

Keywords: spinors, Dirac fields, Majorana fields, two-component fields, rotations and boosts, P, C, T

1. Introduction

A four-component, spin-one-half field invented by Dirac describes the guarks and leptons of the standard model. Dirac fields are therefore of enormous importance in particle physics as well as in nuclear and atomic physics, and in cosmology. Nearly a century has passed since Dirac's original description of spin-one-half fields. One therefore might expect that they would be explained clearly and thoroughly in all modern textbooks on quantum field theory.

Not quite. All modern textbooks on quantum field theory describe to some extent the annihilation operator $a\,(\boldsymbol{p}, s)$, which deletes a particle of momentum $\boldsymbol{p}$ and spin s, and the creation operator $a_c^\dagger(\boldsymbol{p}, s)$, which adds an antiparticle of momentum $\boldsymbol{p}$ and spin s. They all get the 2π's and the phase factors $\mathrm{e}^{\pm \mathrm{i}p\cdot x}$ right and provide for the Dirac field a formula like

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$$\psi_D(x) = \sum_{s=\pm 1/2} \int \frac{d^3p}{(2\pi)^{3/2}} \left[u_D(\boldsymbol{p}, s)\, e^{ip\cdot x}\, a(\boldsymbol{p}, s) + v_D(\boldsymbol{p}, s)\, e^{-ip\cdot x}\, a_c^\dagger(\boldsymbol{p}, s) \right] \tag{1}$$

in which $p \cdot x = \boldsymbol{p} \cdot \boldsymbol{x} - p^0 t$, $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$, $\hbar = c = 1$, and Dirac's index D runs from 1 to 4. But three of the leading textbooks on quantum field theory [1–3] give incorrect formulas for the spinors $v_D(\boldsymbol{p}, s)$ that multiply creation operators. And only five [4–8] of the 15 leading textbooks give explicit, correct formulas for the spinors.

The only book that fully explains what spinors are and that derives formulas for them is *The Quantum Theory of Fields I* by Steven Weinberg [8]. His treatment of this and other topics is so deep and so general, however, that he had to skip many intermediate steps to avoid having his book run to several thousand pages. The paper [9] by Peter Cahill and me fills in some of these steps.

My purpose in this paper is to point out that one may use the Dirac equation and elementary quantum mechanics, specifically how the states of a spin-one-half system of momentum zero transform under rotations, to derive explicit formulas for the spinors $u_D(\boldsymbol{p}, s)$ and $v_D(\boldsymbol{p}, s)$.

The Dirac equation in momentum space yields the spinors at finite momentum $u_D(\boldsymbol{p}, s)$ and $v_D(\boldsymbol{p}, s)$ once we know the spinors at momentum zero $u_D(\boldsymbol{0}, s)$ and $v_D(\boldsymbol{0}, s)$, but it does not tell us what the spinors are at momentum zero. It merely tells us that the spinors $u_D(\boldsymbol{0}, s)$ are eigenstates of γ^0 with eigenvalue $-i$, and that the spinors $v_D(\boldsymbol{0}, s)$ are eigenstates of γ^0 with eigenvalue i. But the eigenvalues $-i$ and i are degenerate; each has two eigenvectors. It is the way a nonrelativistic spin-one-half system transforms under rotations that tells us both which of the degenerate eigenvectors of γ^0 with eigenvalue $-i$ is $u_D(\boldsymbol{0}, \frac{1}{2})$ and which is $u_D(\boldsymbol{0}, -\frac{1}{2})$, and also which of the degenerate eigenvectors of γ^0 with eigenvalue i is $v_D(\boldsymbol{0}, \frac{1}{2})$ and which is $v_D(\boldsymbol{0}, -\frac{1}{2})$. The Dirac equation then immediately gives us the spinors at finite momentum $u_D(\boldsymbol{p}, s)$ and $v_D(\boldsymbol{p}, s)$. This derivation is the simplest one I know of and the one that most reflects the way spin-one-half systems behave under rotations.

Some of the notation used in this paper and elsewhere is described in section 2. Section 3 reviews how the Pauli matrices represent rotations of nonrelativistic, spin-one-half systems. This knowledge is applied in section 4 to two nonrelativistic, spin-one-half systems: the momentum-zero annihilation operators $a(\boldsymbol{0}, s)$ for $s = \pm\frac{1}{2}$ and the momentum-zero antiparticle creation operators $a_c^\dagger(\boldsymbol{0}, s)$. This understanding of how spin-one-half creation and annihilation operators of momentum zero transform under rotations is used in section 5 to explain how rotations about the z axis transform the two 2-component fields that make up a 4-component Dirac field and to determine their momentum-zero, 2-component spinors $u_\alpha(\boldsymbol{0}, s)$ and $v_\alpha(\boldsymbol{0}, s)$ for $\alpha = 1, 2$. The Dirac equation in momentum space at momentum zero is used in section 6 to determine how the four momentum-zero 2-component spinors $u_\alpha(\boldsymbol{0}, s)$ and $v_\alpha(\boldsymbol{0}, s)$ are combined into two momentum-zero, 4-component Dirac spinors $u_D(\boldsymbol{0}, s)$ and $v_D(\boldsymbol{0}, s)$. Section 7 shows how to use the Dirac equation in momentum space to determine the 4-component Dirac spinors $u_D(\boldsymbol{p}, s)$ and $v_D(\boldsymbol{p}, s)$ at momentum $\boldsymbol{p}$ from the spinors $u_D(\boldsymbol{0}, s)$ and $v_D(\boldsymbol{0}, s)$ at momentum zero. These Sections are aimed at students who have had a good course in quantum mechanics at the level of the book *Modern Quantum Mechanics* by Sakurai [10].

The two 2-component fields that make up a 4-component Dirac field transform the same way under rotations but differently under Lorentz boosts. Section 8 explains why these 2-component fields are called left handed and right handed and why at high momentum left-handed fields annihilate particles whose momenta are antiparallel to their spins and create antiparticles whose momenta are parallel to their spins while right-handed fields annihilate particles whose momenta are parallel to their spins and create antiparticles whose momenta are antiparallel to their spins. This section and the three appendices that follow it are at the graduate level.

Appendix A uses the properties of spinors as explained in sections 6 and 7 to derive the behavior of a Dirac field under parity, charge conjugation, and time reversal. Appendix B shows that the requirement that a Dirac field transform properly under Lorentz transformations determines the spinors. Appendix C describes Majorana and 2-component fields.

2. Notation

This section describes some symbols used in this paper and others that are merely worth knowing.

Spin one-half. When measured along any axis, the spin of a spin-one-half particle is $\pm\hbar/2$.

Units. $\hbar = c = 1$.

Kronecker delta. The Kronecker delta is

$$\delta_{s,s'} = \begin{cases} 1 & \text{if} \quad s = s' \\ 0 & \text{if} \quad s \neq s' \end{cases} \quad . \tag{2}$$

Metric. The metric of flat spacetime is the 4×4 diagonal matrix η

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Its elements are $\eta_{00} = \eta^{00} = -1$, and $\eta_{ik} = \eta^{ik} = \delta_{ik}$ for $i, k = 1, 2, 3$ with $\eta_{i0} = \eta^{i0} = \eta_{0i} = \eta^{0i} = 0$. Many authors use $-\eta$ instead.

4-vectors. The 4-vectors of position and momentum are $x = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x})$ and $p = (p^0, p^1, p^2, p^3) = (E, \boldsymbol{p})$. With lowered indexes, they are $(x_0, x_1, x_2, x_3) = (-t, \boldsymbol{x})$ and $(p_0, p_1, p_2, p_3) = (-E, \boldsymbol{p})$.

Summation convention. A repeated index often is meant to be summed over, as in

$$p \cdot x = p^i x_i = p_i x^i = \sum_{i=0}^{3} p_i\, x^i = \boldsymbol{p} \cdot \boldsymbol{x} - p^0 x^0 \quad \text{and} \quad x_i = \eta_{ik}\, x^k. \tag{4}$$

Dirac delta function. The functional $\delta(x - y)$ maps the function $f(x)$ to the number $f(y)$

$$f(y) = \int \mathrm{d}x\, f(x)\, \delta(x - y). \tag{5}$$

Dirac notation. A state that represents a particle of kind n, momentum $\boldsymbol{p}$, and spin s in the z direction called a ket and is written as $|\boldsymbol{p}, s, n\rangle$. The Hermitian adjoint of the ket $|\boldsymbol{p}', s', n'\rangle$ is the bra $\langle \boldsymbol{p}', s', n'|$. Their inner product is $\langle \boldsymbol{p}', s', n'|\boldsymbol{p}, s, n\rangle = \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')\delta_{s,s'}\delta_{nn'}$.

$\mathbf{d^3}\boldsymbol{p}$. The differential $\mathrm{d}^3 p$ is $\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$ or equivalently $\mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3$.

Annihilation operator. The annihilation operators $a(\boldsymbol{p}, s)$ and $a_c(\boldsymbol{p}, s)$ respectively delete from a state either a particle or an antiparticle of momentum $\boldsymbol{p}$ and spin s in the z direction. The energy of the particle or antiparticle is $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ where $m > 0$ is the mass of the particle. All the annihilation operators of a field delete particles of the same mass.

Creation operator. The creation operators $a^\dagger(\boldsymbol{p}, s)$ and $a_c^\dagger(\boldsymbol{p}, s)$ respectively add to a state either a particle or an antiparticle of momentum $\boldsymbol{p}$ and spin s in the z direction. The energy of the particle or antiparticle is $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ where $m > 0$ is the mass of the particle. All the creation operators of a field add particles of the same mass.

Spinors. The spinor $u_D(\boldsymbol{p}, s)$ has four components, $D = 1, 2, 3, 4$. It is the coefficient of the annihilation operator $a(\boldsymbol{p}, s)$ in the Fourier expansion (1) of a Dirac field. The spinor $v_D(\boldsymbol{p}, s)$ has four components, $D = 1, 2, 3, 4$. It is the coefficient of the creation operator $a^\dagger(\boldsymbol{p}, s)$ in the Fourier expansion (1) of a Dirac field.

Dirac field. The Dirac field (1) has four components $D = 1, 2, 3, 4$. It is composed of two 2-component fields that transform the same way under rotations but differently under Lorentz boosts.

Momentum operator. The Hermitian 3-vector $\boldsymbol{P}$ is the momentum operator.

Spin operator. The Hermitian 3-vector $\boldsymbol{S}$ is the spin operator.

Orbital angular-momentum operator. The Hermitian 3-vector $\boldsymbol{L}$ is the orbital angular-momentum operator.

Angular-momentum operator. The angular-momentum operator is $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

Ket $|\boldsymbol{p}, s\rangle$. The ket $|\boldsymbol{p}, s\rangle$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue $\boldsymbol{p}$ and of S_z with eigenvalue s, so $\boldsymbol{P}|\boldsymbol{p}, s\rangle = \boldsymbol{p}|\boldsymbol{p}, s\rangle$ and $S_z|\boldsymbol{p}, s\rangle = s|\boldsymbol{p}, s\rangle$.

Rotations. An active, right-handed rotation $R_{\boldsymbol{\theta}}$ is a 3×3 orthogonal matrix that rotates 3-vectors by θ radians about the axis $\hat{\boldsymbol{\theta}}$ in the right-handed way. For instance, a right-handed rotation by $\pi/2$ about the $\hat{\boldsymbol{y}}$ axis takes the 3-vector $\hat{\boldsymbol{z}}$ into $\hat{\boldsymbol{x}}$.

Rotation operator. The operator that represents an active, right-handed rotation of θ radians about the axis $\hat{\boldsymbol{\theta}}$ is the unitary operator $U(R_{\boldsymbol{\theta}}) = \exp(-\mathrm{i}\boldsymbol{\theta} \cdot \boldsymbol{J})$. So the operator that represents an active, right-handed rotation of θ radians about the axis $\hat{\boldsymbol{z}}$ is the unitary operator $U(R_{\theta\hat{\boldsymbol{z}}}) = \exp(-\mathrm{i}\theta J_z)$.

Pauli matrices. The Pauli matrices are the three 2×2 Hermitian matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

They satisfy $\sigma_i \sigma_j = \delta_{ij} + \mathrm{i}\epsilon_{\mathrm{ijk}}\sigma_k$.

Commutators and anticommutators. The commutator of two operators A and B is $[A, B] \equiv AB - BA$; their anticommutator is $\{A, B\} \equiv AB + BA$.

Gamma matrices. Dirac's gamma matrices are any set of four 4×4 matrices γ^i that satisfy the anticommutation relation $\{\gamma^i, \gamma^k\} \equiv \gamma^i\gamma^k + \gamma^k\gamma^i = 2\eta^{ik}$. The ones used in this paper are those of Weyl and Weinberg [11]

$$\gamma^0 = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = -\mathrm{i}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{7}$$

They have an extra factor of i because I use the metric $(-1, +1, +1, +1)$. Every nonsingular 4×4 matrix S yields another set of gamma matrices $\gamma'^i = S\gamma^i S^{-1}$ that obey the condition $\{\gamma'^i, \gamma'^k\} = 2\eta^{ik}$.

Equal-time anticommutation relations. A Dirac field obeys the equal-time anticommutation relations $\{\psi_D(t, \boldsymbol{x}), \psi_{D'}(t, \boldsymbol{y})\} = 0$ and $\{\psi_D(t, \boldsymbol{x}), \psi_{D'}^\dagger(t, \boldsymbol{y})\} = \delta(\boldsymbol{x} - \boldsymbol{y})\,\delta_{D,D'}$.

3. Rotations of states of zero momentum and spin one-half

The two states of a spin-one-half particle of momentum zero transform under rotations like the spin-one-half systems of nonrelativistic quantum mechanics as discussed in, for example, chapter 3 of the book by Sakurai [10]. This section reviews some details that will be used in later sections.

A spin-one-half particle of momentum $\boldsymbol{p} = \boldsymbol{0}$ and spin $s = \pm\frac{1}{2}$ in the z direction is represented by a state $|\boldsymbol{0}, s\rangle$ that is an eigenstate of the momentum operator $\boldsymbol{P}$ and of the z component S_z of the spin part $\boldsymbol{S}$ of the angular-momentum operator $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ with eigenvalues $\boldsymbol{0}$ and s

$$\boldsymbol{P}|\boldsymbol{0}, s\rangle = \boldsymbol{0} \quad \text{and} \quad S_z|\boldsymbol{0}, s\rangle = s|\boldsymbol{0}, s\rangle. \tag{8}$$

The operator $\boldsymbol{L}$ is the orbital part of the angular-momentum operator $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

The spin operator $\boldsymbol{S}$ is represented in terms of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

as

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma} = \frac{\boldsymbol{\sigma}}{2} \quad \text{or} \quad \langle s'|S_i|s\rangle = \frac{1}{2}(S_i)_{s',s}. \tag{10}$$

A right-handed, active rotation $R_{\boldsymbol{\theta}}$ of $\theta = |\boldsymbol{\theta}|$ radians about the axis $\hat{\boldsymbol{\theta}}$ is represented by the unitary operator $\mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J}} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot(\boldsymbol{L}+\boldsymbol{S})}$. When the momentum is zero, this rotation leaves the momentum $\boldsymbol{p} = \boldsymbol{0}$ unchanged. So the total angular-momentum operator $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ and the spin angular-momentum operator $\boldsymbol{S}$ have the same effect on a state $|\boldsymbol{0}, s\rangle$ of momentum zero. They both rotate the state $|\boldsymbol{0}, s\rangle$ to a linear combination of the two spin states $|\boldsymbol{0}, \pm\frac{1}{2}\rangle$

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J}}|\boldsymbol{0}, s\rangle &= \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{S}}|\boldsymbol{0}, s\rangle = \sum_{s'=-\frac{1}{2}}^{\frac{1}{2}} |\boldsymbol{0}, s'\rangle\langle s'|\mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{S}}|s\rangle \\ &= \sum_{s'=-\frac{1}{2}}^{\frac{1}{2}} \left[\mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}\right]_{s's} |\boldsymbol{0}, s'\rangle = \sum_{s'=-\frac{1}{2}}^{\frac{1}{2}} D_{s's}(R_{\boldsymbol{\theta}})\, |\boldsymbol{0}, s'\rangle \end{aligned} \tag{11}$$

in which

$$I = \sum_{s'=-\frac{1}{2}}^{\frac{1}{2}} |s'\rangle\langle s'| \tag{12}$$

is the identity operator and

$$D_{s's}(R_{\boldsymbol{\theta}}) = \langle s'|\mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{S}}|s\rangle = \left[\mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}\right]_{s's} \tag{13}$$

is the 2×2 unitary matrix that represents the rotation $R_{\boldsymbol{\theta}}$.

The identity

$$\sigma_i\sigma_j = \delta_{ij} + i\sum_{k=1}^{3} \epsilon_{ijk}\sigma_k \tag{14}$$

implies that $(\boldsymbol{\theta}\cdot\boldsymbol{\sigma})^2 = \boldsymbol{\theta}\cdot\boldsymbol{\theta}$ which one may use to show that the $SU(2)$ matrix $D_{s's}(R_{\boldsymbol{\theta}}) = \left[\mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}\right]_{s's}$ is a trigonometric combination of the Pauli matrices

$$D_{s's}(R_{\boldsymbol{\theta}}) = \delta_{s's}\,\cos(\theta/2) - \mathrm{i}(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})_{s's}\,\sin(\theta/2). \tag{15}$$

For example, a rotation about the $\hat{z}$ axis by angle θ is represented by the 2×2 matrix

$$D(R_{\theta\hat{\mathbf{z}}}) = \mathrm{e}^{-\mathrm{i}\theta\sigma_z/2} = I\,\cos(\theta/2) - \mathrm{i}\sigma_z\,\sin(\theta/2) = \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\theta} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\theta}\end{pmatrix}, \tag{16}$$

and so

$$\mathrm{e}^{-\mathrm{i}\theta J_z}|\mathbf{0},s\rangle = \mathrm{e}^{-\mathrm{i}\theta S_z}|\mathbf{0},s\rangle = \mathrm{e}^{-\mathrm{i}\theta s/2}|\mathbf{0},s\rangle. \tag{17}$$

Similarly, a rotation by π about the y axis is represented by the 2×2 matrix

$$D(R_{\pi\hat{\mathbf{y}}}) = \mathrm{e}^{-\mathrm{i}\pi\sigma_y/2} = I\,\cos(\pi/2) - \mathrm{i}\sigma_y\,\sin(\pi/2) = -\mathrm{i}\sigma_y = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{18}$$

and so

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\pi J_y}|\mathbf{0},\tfrac{1}{2}\rangle &= \mathrm{e}^{-\mathrm{i}\pi S_y}|\mathbf{0},\tfrac{1}{2}\rangle = |\mathbf{0},-\tfrac{1}{2}\rangle\\ \mathrm{e}^{-\mathrm{i}\pi J_y}|\mathbf{0},\tfrac{1}{2}\rangle &= \mathrm{e}^{-\mathrm{i}\pi S_y}|\mathbf{0},-\tfrac{1}{2}\rangle = -|\mathbf{0},\tfrac{1}{2}\rangle,\end{aligned} \tag{19}$$

while a rotation by π about the x axis is represented by the 2×2 matrix

$$D(R_{\pi\hat{\mathbf{x}}}) = \mathrm{e}^{-\mathrm{i}\pi\sigma_x/2} = I\,\cos(\pi/2) - \mathrm{i}\sigma_x\,\sin(\pi/2) = -\mathrm{i}\sigma_x = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} \tag{20}$$

which implies that

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\pi J_x}|\mathbf{0},\tfrac{1}{2}\rangle &= \mathrm{e}^{-\mathrm{i}\pi S_x}|\mathbf{0},\tfrac{1}{2}\rangle = -\mathrm{i}|\mathbf{0},-\tfrac{1}{2}\rangle\\ \mathrm{e}^{-\mathrm{i}\pi J_x}|\mathbf{0},\tfrac{1}{2}\rangle &= \mathrm{e}^{-\mathrm{i}\pi S_x}|\mathbf{0},-\tfrac{1}{2}\rangle = -\mathrm{i}|\mathbf{0},\tfrac{1}{2}\rangle.\end{aligned} \tag{21}$$

4. Rotations of spin-one-half creation and annihilation operators of momentum zero

The creation and annihilation operators of a spin-one-half particle of momentum zero transform under rotations like the two states of the spin-one-half systems reviewed in section 3.

The state $|\mathbf{0},s\rangle$ of a spin-one-half particle of momentum zero and spin $s=\pm\frac{1}{2}$ is formed from the vacuum state $|0\rangle$ by the creation operator $a^\dagger(\mathbf{0},s)$

$$a^\dagger(\mathbf{0},s)\,|0\rangle = |\mathbf{0},s\rangle. \tag{22}$$

Similarly, the antiparticle creation operator $a_{\mathrm{c}}^\dagger(\mathbf{0},s)$ adds an antiparticle of momentum $\mathbf{0}$ and spin s to the vacuum state

$$a_{\mathrm{c}}^\dagger(\mathbf{0},s)\,|0\rangle = |\mathbf{0},s\rangle_{\mathrm{c}}. \tag{23}$$

We have seen (17) that a right-handed rotation about the $\hat{z}$ axis by angle θ takes the state $|\mathbf{0}, s\rangle$ to $\mathrm{e}^{-\mathrm{i}s\theta}|\mathbf{0}, s\rangle$, so we find

$$\mathrm{e}^{-\mathrm{i}\theta J_z}\, a^\dagger(\mathbf{0}, s)\,|0\rangle = \mathrm{e}^{-\mathrm{i}\theta J_z}\,|\mathbf{0}, s\rangle = \mathrm{e}^{-\mathrm{i}s\theta}\,|\mathbf{0}, s\rangle = \mathrm{e}^{-\mathrm{i}s\theta}\, a^\dagger(\mathbf{0}, s)\,|0\rangle \tag{24}$$

and similarly

$$\mathrm{e}^{-\mathrm{i}\theta J_z}\, a_\mathrm{c}^\dagger(\mathbf{0}, s)\,|0\rangle = \mathrm{e}^{-\mathrm{i}\theta J_z}\,|\mathbf{0}, s\rangle_\mathrm{c} = \mathrm{e}^{-\mathrm{i}s\theta}\,|\mathbf{0}, s\rangle_\mathrm{c} = \mathrm{e}^{-\mathrm{i}s\theta}\, a_\mathrm{c}^\dagger(\mathbf{0}, s)\,|0\rangle_\mathrm{c}. \tag{25}$$

In the flat space-time of quantum field theory, the vacuum is invariant under rotations, so

$$\mathrm{e}^{\mathrm{i}\theta J_z}\,|0\rangle = |0\rangle. \tag{26}$$

Thus we can rewrite equations (24) and (25) as

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\theta J_z}\, a^\dagger(\mathbf{0}, s)\,\mathrm{e}^{\mathrm{i}\theta J_z}\,|0\rangle &= \mathrm{e}^{-\mathrm{i}s\theta}\, a^\dagger(\mathbf{0}, s)\,|0\rangle \\ \mathrm{e}^{-\mathrm{i}\theta J_z}\, a_\mathrm{c}^\dagger(\mathbf{0}, s)\,\mathrm{e}^{\mathrm{i}\theta J_z}\,|0\rangle &= \mathrm{e}^{-\mathrm{i}s\theta}\, a_\mathrm{c}^\dagger(\mathbf{0}, s)\,|0\rangle. \end{aligned} \tag{27}$$

Replacing the vacuum state $|0\rangle$ by an arbitrary state $|\psi\rangle$ and repeating the last few steps, we see that a z rotation changes the creation operators $a^\dagger(\mathbf{0}, s)$ and $a_\mathrm{c}^\dagger(\mathbf{0}, s)$ to

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\theta J_z}\, a^\dagger(\mathbf{0}, s)\,\mathrm{e}^{\mathrm{i}\theta J_z} &= \mathrm{e}^{-\mathrm{i}s\theta}\, a^\dagger(\mathbf{0}, s) \\ \mathrm{e}^{-\mathrm{i}\theta J_z}\, a_\mathrm{c}^\dagger(\mathbf{0}, s)\,\mathrm{e}^{\mathrm{i}\theta J_z} &= \mathrm{e}^{-\mathrm{i}s\theta}\, a_\mathrm{c}^\dagger(\mathbf{0}, s). \end{aligned} \tag{28}$$

The adjoints of these equations are

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\theta J_z} a(\mathbf{0}, s)\mathrm{e}^{\mathrm{i}\theta J_z} &= \mathrm{e}^{\mathrm{i}s\theta} a(\mathbf{0}, s) \\ \mathrm{e}^{-\mathrm{i}\theta J_z} a_\mathrm{c}(\mathbf{0}, s)\mathrm{e}^{\mathrm{i}\theta J_z} &= \mathrm{e}^{\mathrm{i}s\theta} a_\mathrm{c}(\mathbf{0}, s) \end{aligned} \tag{29}$$

Under rotations, creation and annihilation operators transform with opposite phases. That's why the $u(\mathbf{0}, s)$ spinors that multiply annihilation operators are and different from the $v(\mathbf{0}, s)$ spinors that multiply creation operators.

If instead of the z axis, the rotation is about an arbitrary axis $\hat{\boldsymbol{\theta}}$, then the equations that replace (28) and (29) are

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J}} a(\mathbf{0}, s)\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J}} &= \sum_{s'} D^*_{s's}(R_{\boldsymbol{\theta}})\ a(\mathbf{0}, s') = \sum_{s'} D^{-1}_{ss'}(R_{\boldsymbol{\theta}})\ a(\mathbf{0}, s') \\ \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J}} a_\mathrm{c}^\dagger(\mathbf{0}, s)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J}} &= \sum_{s'} D_{s's}(R_{\boldsymbol{\theta}})\, a_\mathrm{c}^\dagger(\mathbf{0}, s') = \sum_{s'} D^{*-1}_{ss'}(R_{\boldsymbol{\theta}})\, a_\mathrm{c}^\dagger(\mathbf{0}, s'). \end{aligned} \tag{30}$$

In particular, a rotation by π about the y axis changes $a(\mathbf{0}, s)$ and $a_\mathrm{c}^\dagger(\mathbf{0}, s)$ to

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\pi J_y} a(\mathbf{0}, s)\mathrm{e}^{\mathrm{i}\pi J_y} &= \sum_{s'} D^*_{s's}(R_{\pi\hat{\mathbf{y}}})\ a(\mathbf{0}, s') = \sum_{s'} D^{-1}_{ss'}(R_{\pi\hat{\mathbf{y}}})\ a(\mathbf{0}, s') \\ \mathrm{e}^{-\mathrm{i}\pi J_y} a_\mathrm{c}^\dagger(\mathbf{0}, s)\,\mathrm{e}^{\mathrm{i}\pi J_y} &= \sum_{s'} D_{s's}(R_{\pi\hat{\mathbf{y}}})\, a_\mathrm{c}^\dagger(\mathbf{0}, s') = \sum_{s'} D^{*-1}_{ss'}(R_{\pi\hat{\mathbf{y}}})\, a_\mathrm{c}^\dagger(\mathbf{0}, s'). \end{aligned} \tag{31}$$

That is,

$$e^{-i\pi J_y}\begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix} e^{i\pi J_y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} a(\mathbf{0},-\frac{1}{2}) \\ -a(\mathbf{0},\frac{1}{2}) \end{pmatrix} \tag{32}$$

and

$$e^{-i\pi J_y}\begin{pmatrix} a_c^\dagger(\mathbf{0},\frac{1}{2}) \\ a_c^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix} e^{i\pi J_y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_c^\dagger(\mathbf{0},\frac{1}{2}) \\ a_c^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} a_c^\dagger(\mathbf{0},-\frac{1}{2}) \\ -a_c^\dagger(\mathbf{0},\frac{1}{2}) \end{pmatrix}. \tag{33}$$

And a rotation by π about the x axis changes $a(\mathbf{0}, s)$ and $a_c^\dagger(\mathbf{0}, s)$ to

$$\begin{aligned} e^{-i\pi J_x} a(\mathbf{0},s) e^{i\pi J_x} &= \sum_{s'} D^*_{s's}(R_{\pi\hat{\mathbf{x}}})\ a(\mathbf{0},s') = \sum_{s'} D^{-1}_{ss'}(R_{\pi\hat{\mathbf{x}}})\ a(\mathbf{0},s') \\ e^{-i\pi J_x} a_c^\dagger(\mathbf{0},s)\, e^{i\pi J_x} &= \sum_{s'} D_{s's}(R_{\pi\hat{\mathbf{x}}})\, a_c^\dagger(\mathbf{0},s') = \sum_{s'} D^{*-1}_{ss'}(R_{\pi\hat{\mathbf{x}}})\, a_c^\dagger(\mathbf{0},s'). \end{aligned} \tag{34}$$

That is,

$$e^{-i\pi J_x}\begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix} e^{i\pi J_x} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} i a(\mathbf{0},-\frac{1}{2}) \\ i a(\mathbf{0},\frac{1}{2}) \end{pmatrix} \tag{35}$$

and

$$e^{-i\pi J_x}\begin{pmatrix} a^\dagger(\mathbf{0},\frac{1}{2}) \\ a^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix} e^{i\pi J_x} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} a^\dagger(\mathbf{0},\frac{1}{2}) \\ a^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} -i a^\dagger(\mathbf{0},-\frac{1}{2}) \\ -i a^\dagger(\mathbf{0},\frac{1}{2}). \end{pmatrix} \tag{36}$$

5. Rotations of two-component spin-one-half fields

A Dirac field is made of two 2-component spin-one-half fields that transform the same way under rotations but differently under Lorentz boosts. This section explains how these 2-component spin-one-half fields transform under rotations and derives formulas for their 2-component spinors at momentum zero.

Like the upper or lower two components of the Fourier expansion (1) of a four-component spin-one-half field, the Fourier expansion of a two-component spin-one-half field is

$$\psi_\alpha(x) = \sum_{s=\pm 1/2} \int \frac{d^3p}{(2\pi)^{3/2}} \left[u_\alpha(\boldsymbol{p},s)\, e^{ip\cdot x}\, a(\boldsymbol{p},s) + v_\alpha(\boldsymbol{p},s)\, e^{-ip\cdot x}\, a_c^\dagger(\boldsymbol{p},s) \right] \tag{37}$$

in which $\alpha = 1, 2$ and $p \cdot x = \boldsymbol{p} \cdot \boldsymbol{x} - p^0 t$ with $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$.

Under a rotation $R_{\boldsymbol{\theta}}$ represented by the matrix $D(R_{\boldsymbol{\theta}}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}$ of equation (15), a 2-component spin-one-half field $\psi_\alpha(x)$ transforms as

$$U(R_{\boldsymbol{\theta}})\psi_\alpha(x)U^{-1}(R_{\boldsymbol{\theta}}) = \mathrm{e}^{-\mathrm{i}\pi J_x}\psi_\alpha(x)\mathrm{e}^{\mathrm{i}\pi J_x} = \sum_{\beta=1}^{2} D_{\alpha\beta}(R_{\boldsymbol{\theta}}^{-1})\,\psi_\beta(R_{\boldsymbol{\theta}}x)$$
$$= \sum_{\beta=1}^{2}\left(\mathrm{e}^{\mathrm{i}\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\right)_{\alpha\beta}\psi_\beta(R_{\boldsymbol{\theta}}x). \tag{38}$$

This equation may be taken as part of the definition of a spin-one-half field, another part being how the field transforms (100) under Lorentz transformations that are not simple rotations. But because the effects of rotations can be confusing, I will sketch one aspect of this equation (38) by writing its mean value in some state $|W\rangle$ of the world as

$$\langle W|U(R_{\boldsymbol{\theta}})\psi(x)U^{-1}(R_{\boldsymbol{\theta}})|W\rangle = \langle R^{-1}W|\psi(x)|R^{-1}W\rangle = D(R^{-1})\langle W|\psi(Rx)|W\rangle. \tag{39}$$

Apart from the matrix $D(R^{-1})$, this equation says that the mean value of the field at x in a world rotated by R^{-1} is the same as its mean value at Rx in an unrotated world. But the field ψ is a vector whose mean value depends upon the world. So in a world rotated by R^{-1}, its mean value is rotated by R^{-1}.

The matrix $D(R_{\theta\hat{z}})$ that represents a rotation about the z axis (16) is diagonal because σ_z is diagonal

$$D(R_{\theta\hat{\boldsymbol{z}}}) = \mathrm{e}^{-\mathrm{i}\theta\sigma_z} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}. \tag{40}$$

So under a rotation about the $\boldsymbol{\theta} = \theta\hat{\boldsymbol{z}}$ axis, the field $\psi_\alpha(x)$ transforms as

$$U(R_{\theta\hat{\boldsymbol{z}}})\psi_\alpha(x)U^{-1}(R_{\theta\hat{\boldsymbol{z}}}) = \mathrm{e}^{-\mathrm{i}\theta J_z}\psi_\alpha(x)\,\mathrm{e}^{\mathrm{i}\theta J_z} = \sum_{\beta=1}^{2}\ \exp\left(\mathrm{i}\frac{1}{2}\theta\sigma_z\right)_{\alpha\beta}\psi_\beta(R_{\theta\hat{\boldsymbol{z}}}x). \tag{41}$$

We are after the spinors at momentum zero, so we need pay attention only to the momentum-zero part $\psi^0_\alpha(x)(x)$ of the field. We also can distinguish between the annihilator part $\psi^{(0,+)}_\alpha(x)$ and the creator part $\psi^{(0,-)}_\alpha(x)$. Apart from the factors of 2π, the momentum-zero, annihilator part $\psi^{(0,+)}_\alpha(x)$ at $x = 0$ is

$$\begin{pmatrix} \psi_1^{(0,+)}(0) \\ \psi_2^{(0,+)}(0) \end{pmatrix} = \begin{pmatrix} u_1(0,\frac{1}{2}) \\ u_2(0,\frac{1}{2}) \end{pmatrix} a(0,\frac{1}{2}) + \begin{pmatrix} u_1(0,-\frac{1}{2}) \\ u_2(0,-\frac{1}{2}) \end{pmatrix} a(0,-\frac{1}{2}). \tag{42}$$

The formula (38) for how the field $\psi_\alpha(x)$ transforms under the rotation $R_{\boldsymbol{\theta}}$ simplifies for the momentum-zero, annihilator part $\psi^{(0,+)}_\alpha(x)$ at $x = 0$ to

$$\mathrm{e}^{\mathrm{i}\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\,\psi^{(0,+)}(0) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi^{(0,+)}(0)\mathrm{e}^{\mathrm{i}\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}. \tag{43}$$

For a rotation about the z axis, this is more explicitly

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \left[\begin{pmatrix} u_1(0,\frac{1}{2}) \\ u_2(0,\frac{1}{2}) \end{pmatrix} a(0,\frac{1}{2}) + \begin{pmatrix} u_1(0,-\frac{1}{2}) \\ u_2(0,-\frac{1}{2}) \end{pmatrix} a(0,-\frac{1}{2}) \right]$$
$$= \begin{pmatrix} e^{i\theta/2}\,u_1(0,\frac{1}{2}) \\ e^{-i\theta/2}\,u_2(0,\frac{1}{2}) \end{pmatrix} a(0,\frac{1}{2}) + \begin{pmatrix} e^{i\theta/2}\,u_1(0,-\frac{1}{2}) \\ e^{-i\theta/2}\,u_2(0,-\frac{1}{2}) \end{pmatrix} a(0,-\frac{1}{2})$$
$$= \begin{pmatrix} u_1(0,\frac{1}{2}) \\ u_2(0,\frac{1}{2}) \end{pmatrix} e^{i\theta/2}\,a(0,\frac{1}{2}) + \begin{pmatrix} u_1(0,-\frac{1}{2}) \\ u_2(0,-\frac{1}{2}) \end{pmatrix} e^{-i\theta/2}\,a(0,-\frac{1}{2}). \tag{44}$$

We see that the momentum-zero 2-spinors are

$$u(0,\frac{1}{2}) = c_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u(0,-\frac{1}{2}) = c_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{45}$$

as one would have expected without the present derivation.

Apart from the factors of 2π, the momentum-zero, creator part $\psi_\alpha^{(0,-)}(x)$ at $x = 0$ is

$$\begin{pmatrix} \psi_1^{(0,-)}(0) \\ \psi_2^{(0,-)}(0) \end{pmatrix} = \begin{pmatrix} v_1(0,\frac{1}{2}) \\ v_2(0,\frac{1}{2}) \end{pmatrix} a_c^\dagger(0,\frac{1}{2}) + \begin{pmatrix} v_1(0,-\frac{1}{2}) \\ v_2(0,-\frac{1}{2}) \end{pmatrix} a_c^\dagger(0,-\frac{1}{2}). \tag{46}$$

The formula (38) for how the field $\psi_\alpha(x)$ transforms under the rotation $R_{\boldsymbol{\theta}}$ simplifies for the momentum-zero, creator part $\psi_\alpha^{(0,-)}(x)$ at $x = 0$ to

$$e^{i\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\,\psi^{(0,-)}(0) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}}\psi^{(0,-)}(0)e^{i\boldsymbol{\theta}\cdot\boldsymbol{J}}. \tag{47}$$

For a rotation about the z axis, this is more explicitly

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \left[\begin{pmatrix} v_1(0,\frac{1}{2}) \\ v_2(0,\frac{1}{2}) \end{pmatrix} a_c^\dagger(0,\frac{1}{2}) + \begin{pmatrix} v_1(0,-\frac{1}{2}) \\ v_2(0,-\frac{1}{2}) \end{pmatrix} a_c^\dagger(0,-\frac{1}{2}) \right]$$
$$= \begin{pmatrix} e^{i\theta/2}\,v_1(0,\frac{1}{2}) \\ e^{-i\theta/2}\,v_2(0,\frac{1}{2}) \end{pmatrix} a_c^\dagger(0,\frac{1}{2}) + \begin{pmatrix} e^{i\theta/2}\,v_1(0,-\frac{1}{2}) \\ e^{-i\theta/2}\,v_2(0,-\frac{1}{2}) \end{pmatrix} a_c^\dagger(0,-\frac{1}{2})$$
$$= \begin{pmatrix} v_1(0,\frac{1}{2}) \\ v_2(0,\frac{1}{2}) \end{pmatrix} e^{-i\theta/2}\,a_c^\dagger(0,\frac{1}{2}) + \begin{pmatrix} v_1(0,-\frac{1}{2}) \\ v_2(0,-\frac{1}{2}) \end{pmatrix} e^{i\theta/2}\,a_c^\dagger(0,-\frac{1}{2}). \tag{48}$$

We see that the momentum-zero 2-spinors are

$$v(0,\frac{1}{2}) = d_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v(0,-\frac{1}{2}) = d_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{49}$$

which may surprise us. These formulas are missed in three leading textbooks [1–3].

We use our knowledge (32) of how annihilation operators respond to a rotation (18) by angle π about the y axis. We use the matrix (18) that represents a rotation by π about y axis

$$\mathrm{e}^{\mathrm{i}\pi\sigma_y/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{50}$$

and our rule (57) for how the annihilator part of the momentum-zero field transforms

$$\mathrm{e}^{-\mathrm{i}\pi J_y/2}\psi^{(0,+)}(0)\mathrm{e}^{\mathrm{i}\pi J_y/2} = \mathrm{e}^{\mathrm{i}\pi\sigma_y/2}\,\psi^{(0,+)}(0), \tag{51}$$

which in more detail (32) is

$$\mathrm{e}^{-\mathrm{i}\pi J_y}\begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix}\mathrm{e}^{\mathrm{i}\pi J_y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} a(\mathbf{0},-\frac{1}{2}) \\ -a(\mathbf{0},\frac{1}{2}) \end{pmatrix} \tag{52}$$

to infer that

$$\begin{aligned}
&\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left[c_+\begin{pmatrix}1\\0\end{pmatrix}a(0,\frac{1}{2}) + c_-\begin{pmatrix}0\\1\end{pmatrix}a(0,-\frac{1}{2})\right] \\
&\quad = c_+\begin{pmatrix}0\\-1\end{pmatrix}a(0,\frac{1}{2}) + c_-\begin{pmatrix}1\\0\end{pmatrix}a(0,-\frac{1}{2}) \\
&\quad = \mathrm{e}^{-\mathrm{i}\pi J_y}\left[c_+\begin{pmatrix}1\\0\end{pmatrix}a(0,\frac{1}{2}) + c_-\begin{pmatrix}0\\1\end{pmatrix}a(0,-\frac{1}{2})\right]\mathrm{e}^{\mathrm{i}\pi J_y} \\
&\quad = c_+\begin{pmatrix}1\\0\end{pmatrix}a(0,-\frac{1}{2}) - c_-\begin{pmatrix}0\\1\end{pmatrix}a(0,\frac{1}{2}).
\end{aligned} \tag{53}$$

So now we know that $c_- = c_+$, as may have been anticipated.

Under the same y rotation by π, the creation operators transform as (33)

$$\mathrm{e}^{-\mathrm{i}\pi J_y}\begin{pmatrix} a_{\mathrm{c}}^\dagger(\mathbf{0},\frac{1}{2}) \\ a_{\mathrm{c}}^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix}\mathrm{e}^{\mathrm{i}\pi J_y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_{\mathrm{c}}^\dagger(\mathbf{0},\frac{1}{2}) \\ a_{\mathrm{c}}^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} a_{\mathrm{c}}^\dagger(\mathbf{0},-\frac{1}{2}) \\ -a_{\mathrm{c}}^\dagger(\mathbf{0},\frac{1}{2}) \end{pmatrix}, \tag{54}$$

and so the formula (47) for how the zero-momentum part of the creator field transforms under rotations gives us

$$\begin{aligned}
&\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left[d_+\begin{pmatrix}0\\1\end{pmatrix}a_{\mathrm{c}}^\dagger(0,\frac{1}{2}) + d_-\begin{pmatrix}1\\0\end{pmatrix}a_{\mathrm{c}}^\dagger(0,-\frac{1}{2})\right] \\
&\quad = d_+\begin{pmatrix}1\\0\end{pmatrix}a_{\mathrm{c}}^\dagger(0,\frac{1}{2}) + d_-\begin{pmatrix}0\\-1\end{pmatrix}a_{\mathrm{c}}^\dagger(0,-\frac{1}{2}) \\
&\quad = d_+\begin{pmatrix}0\\1\end{pmatrix}a_{\mathrm{c}}^\dagger(0,-\frac{1}{2}) - d_-\begin{pmatrix}1\\0\end{pmatrix}a_{\mathrm{c}}^\dagger(0,\frac{1}{2}).
\end{aligned} \tag{55}$$

So now we know that $d_+ = -d_-$.

Does the x rotation give the same answers? We use the equations

$$e^{i\pi\sigma_x/2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{56}$$

and

$$e^{i\pi\sigma_x/2}\,\psi^{(0,+)}(0) = e^{-i\pi J_x/2}\psi^{(0,+)}(0)e^{i\pi J_x/2}, \tag{57}$$

and

$$e^{-i\pi J_x} \begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix} e^{i\pi J_x} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} a(\mathbf{0},\frac{1}{2}) \\ a(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} ia(\mathbf{0},-\frac{1}{2}) \\ ia(\mathbf{0},\frac{1}{2}) \end{pmatrix} \tag{58}$$

to infer that

$$\begin{aligned} &\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \left[c_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} a(0,\frac{1}{2}) + c_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} a(0,-\frac{1}{2}) \right] \\ &\quad = c_+ \begin{pmatrix} i \\ 0 \end{pmatrix} a(0,\frac{1}{2}) + c_- \begin{pmatrix} 0 \\ i \end{pmatrix} a(0,-\frac{1}{2}) \\ &\quad = e^{-i\pi J_x} \left[c_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} a(0,\frac{1}{2}) + c_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} a(0,-\frac{1}{2}) \right] e^{i\pi J_x} \\ &\quad = c_+ \begin{pmatrix} 0 \\ i \end{pmatrix} a(0,-\frac{1}{2}) + c_- \begin{pmatrix} i \\ 0 \end{pmatrix} a(0,\frac{1}{2}) \end{aligned} \tag{59}$$

which again tells us that $c_- = c_+$.

The creation versions of these equations are

$$e^{-i\pi J_x} \begin{pmatrix} a_c^\dagger(\mathbf{0},\frac{1}{2}) \\ a_c^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix} e^{i\pi J_x} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} a_c^\dagger(\mathbf{0},\frac{1}{2}) \\ a_c^\dagger(\mathbf{0},-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} -ia_c^\dagger(\mathbf{0},-\frac{1}{2}) \\ -ia_c^\dagger(\mathbf{0},\frac{1}{2}) \end{pmatrix}. \tag{60}$$

and

$$\begin{aligned} &\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \left[d_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} a_c^\dagger(0,\frac{1}{2}) + d_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} a_c^\dagger(0,-\frac{1}{2}) \right] \\ &\quad = d_+ \begin{pmatrix} i \\ 0 \end{pmatrix} a_c^\dagger(0,\frac{1}{2}) + d_- \begin{pmatrix} 0 \\ i \end{pmatrix} a_c^\dagger(0,-\frac{1}{2}) \\ &\quad = d_+ \begin{pmatrix} 0 \\ -i \end{pmatrix} a_c^\dagger(0,-\frac{1}{2}) + d_- \begin{pmatrix} -i \\ 0 \end{pmatrix} a_c^\dagger(0,\frac{1}{2}) \end{aligned} \tag{61}$$

which again tell us that $d_+ = -d_-$.

So with $c = c_+ = c_-$, and $d = d_+ = -d_-$, our momentum-zero spinors are

$$\begin{aligned} u(0,\frac{1}{2}) = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad &\text{and} \quad u(0,-\frac{1}{2}) = c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ v(0,\frac{1}{2}) = d \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad &\text{and} \quad v(0,-\frac{1}{2}) = d \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \end{aligned} \tag{62}$$

We have derived these formulas for 2-component spinors from the Pauli matrices and the $SU(2)$ representation of rotations that they provide. These formulas therefore are independent of the choice (7) and (66) of gamma matrices.

6. Four-component spinors at zero momentum

In this Section, we will use the 2-component spinors (62) and the Dirac equation in momentum space at momentum zero to derive formulas for the 4-component spinors $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s)$ at momentum zero.

We make 4-component u spinors by putting two pairs of 2-component u spinors (62) together

$$u(0, \frac{1}{2}) = \begin{pmatrix} u_\ell(0, \frac{1}{2}) \\ u_r(0, \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} c_\ell \\ 0 \\ c_r \\ 0 \end{pmatrix} \quad \text{and} \quad u(0, -\frac{1}{2}) = \begin{pmatrix} u_\ell(0, -\frac{1}{2}) \\ u_r(0, -\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 0 \\ c_\ell \\ 0 \\ c_r \end{pmatrix} \tag{63}$$

and by putting two pairs of 2-component v spinors (62) together

$$v(0, \frac{1}{2}) = \begin{pmatrix} v_\ell(0, \frac{1}{2}) \\ v_r(0, \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 0 \\ d_\ell \\ 0 \\ d_r \end{pmatrix} \quad \text{and} \quad v(0, -\frac{1}{2}) = \begin{pmatrix} v_\ell(0, -\frac{1}{2}) \\ v_r(0, -\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} -d_\ell \\ 0 \\ -d_r \\ 0 \end{pmatrix}. \tag{64}$$

in which c_ℓ, c_r, d_ℓ, and d_r are arbitrary constants. Dirac's equation at momentum zero will tell us that $c_\ell = c_r$ and that $v_\ell = -v_r$.

The labels ℓ and r refer to whether the spinors behave as left-handed or right-handed 2-component spinors under Lorentz boosts. How a Dirac field is made out of a 2-component left-handed field and a 2-component right-handed field is outlined in section 8 and described more fully in references [12, 13].

To narrow down the range of the constants c_ℓ, c_r, d_ℓ, and d_r in the 4-component spinors (63) and (64), we use Dirac's equation

$$(\boldsymbol{\gamma} \cdot \nabla - \gamma^0 \partial_0 + m)\psi(x) = (\gamma^a \partial_a + m)\psi(x) = 0 \tag{65}$$

and his gamma matrices (7)

$$\gamma^0 = -\mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = -\mathrm{i} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{66}$$

in which the σ_i's are the Pauli matrices (9) and the extra factors of i ensure that the anticommutator of two gamma matrices is $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ in which the diagonal of η is $(-1, 1, 1, 1)$.

When the derivative ∂_a in the Dirac equation acts on $u(\boldsymbol{p}, s)\mathrm{e}^{\mathrm{i}p\cdot x}$, it becomes $\mathrm{i}p_a$; when it acts on $v(\boldsymbol{p}, s)\mathrm{e}^{-\mathrm{i}p\cdot x}$, becomes $-\mathrm{i}p_a$. This is the second reason why the spinors u that multiply annihilation operators are different from those v that multiply creation operators.

The Dirac equation in momentum space therefore splits into one equation for u spinors and another for v spinors

$$(\mathrm{i}p_a\gamma^a + m)u(\boldsymbol{p}, s) = 0 \quad \text{and} \quad (-\mathrm{i}p_a\gamma^a + m)v(\boldsymbol{p}, s) = 0. \tag{67}$$

At $p_a = (-m, 0, 0, 0)$, these equations are $(-i\gamma^0 + 1)u(\mathbf{0}, s) = 0$ for $s = \pm\frac{1}{2}$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\ell \\ 0 \\ c_r \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ c_\ell \\ 0 \\ c_r \end{pmatrix} = 0 \tag{68}$$

and $(\mathrm{i}\gamma^0 + 1)v(\mathbf{0}, s) = 0$ for $s = \pm\frac{1}{2}$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ d_\ell \\ 0 \\ d_r \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -d_\ell \\ 0 \\ -d_r \\ 0 \end{pmatrix} = 0. \tag{69}$$

They require that $c_\ell = c_r$ and that $d_\ell = -d_r$. So with $c_\ell = c_r = c$ and $d_\ell = -d_r = d$, our spinors are

$$\begin{aligned} u(0, \frac{1}{2}) &= c \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u(0, -\frac{1}{2}) = c \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \\ v(0, \frac{1}{2}) &= d \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v(0, -\frac{1}{2}) = d \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{70}$$

Choosing the arbitrary relative phases of creation and annihilation operators and the arbitrary phase of the field and normalizing the spinors, we get the zero-momentum spinors derived by Weinberg [8]

$$\begin{aligned} u\left(\mathbf{0}, \frac{1}{2}\right) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u\left(\mathbf{0}, -\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ v\left(\mathbf{0}, \frac{1}{2}\right) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v\left(\mathbf{0}, -\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{71}$$

They obey the parity conditions

$$u(\mathbf{0}, s) = \mathrm{i}\gamma^0 u(\mathbf{0}, s) \quad \text{and} \quad v(\mathbf{0}, s) = -\mathrm{i}\gamma^0 v(\mathbf{0}, s), \tag{72}$$

the charge-conjugation conditions

$$u(\mathbf{0}, s) = \gamma^2 v^*(\mathbf{0}, s) \quad \text{and} \quad v(\mathbf{0}, s) = \gamma^2 u^*(\mathbf{0}, s), \tag{73}$$

and the time-reversal conditions

$$u^*(\mathbf{0}, s) = (-1)^{\frac{1}{2}+s}\gamma^1\gamma^3 u(\mathbf{0}, -s) \quad \text{and} \quad v^*(\mathbf{0}, s) = (-1)^{\frac{1}{2}+s}\gamma^1\gamma^3 v(\mathbf{0}, -s). \tag{74}$$

The spinors (71) are derived in a book by Steven Weinberg [12] and in various articles [14, 15], and they are discussed by Peskin and Schroeder [5] and by Srednicki [4]. But for reasons of space, style, or emphasis, they are not stated explicitly in seven of the leading textbooks on quantum field theory, and they are stated incorrectly in three of them [1–3].

In those three books, it is assumed that spinors merely need to satisfy the Dirac equation (65). But all that Dirac's equation says about the spinors at zero momentum is that the u spinors are eigenstates of γ^0 with eigenvalue $-i$, and that the v spinors are eigenstates of γ^0 with eigenvalue i

$$\gamma^0\, u(\mathbf{0}, s) = -\mathrm{i}\, u(\mathbf{0}, s) \quad \text{and} \quad \gamma^0\, v(\mathbf{0}, s) = \mathrm{i}\, v(\mathbf{0}, s) \tag{75}$$

which recapitulate (67)–(69). But the eigenvalues $-i$ and i are degenerate; each has two eigenvectors. The books [1–3] interchanged the two eigenvectors $v(\mathbf{0}, \frac{1}{2})$ and $v(\mathbf{0}, -\frac{1}{2})$.

Using the wrong spinors leads to Dirac fields that do not transform correctly under rotations, Lorentz transformations, charge conjugation, or time reversal. Such fields can lead to physical results that are incorrect. For instance, the naive spinors [3]

$$\begin{aligned} u_\uparrow &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_\downarrow = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \\ v_\uparrow &= \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \qquad \text{and} \quad v_\downarrow = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \end{aligned} \tag{76}$$

do not obey the charge-conjugation conditions (73) and instead flip the spin

$$u_\downarrow = \gamma^2 v_\uparrow^*, \quad u_\uparrow = \gamma^2 v_\downarrow^*, \quad v_\downarrow = \gamma^2 u_\uparrow^*, \quad \text{and} \quad v_\uparrow = \gamma^2 u_\downarrow^*. \tag{77}$$

Their use leads to the false conclusion that charge conjugation (A4) flips the spin [3]. The naive v spinors (76) also do not obey the time-reversal conditions (74) and instead introduce spurious signs: $v_\uparrow = \gamma^1\gamma^3 v_\downarrow$ and $v_\downarrow = -\gamma^1\gamma^3 v_\uparrow$. So Dirac and Majorana fields made with the naive spinors (76) are mangled under rotations, Lorentz transformations, charge conjugation, and time reversal.

7. Four-component spinors at finite momentum

In this Section, we will use the zero-momentum, 4-component spinors (71) and the Dirac equation in momentum space to derive formulas for the 4-component spinors $u(\boldsymbol{p}, s)$ and $v(\boldsymbol{p}, s)$

at arbitrary momenta. We will find that the u and v spinors differ more at $\boldsymbol{p} \neq \mathbf{0}$ than at $\boldsymbol{p} = \mathbf{0}$.

The Dirac equation (65) tells us how Dirac fields depend upon the coordinate $x = (x^0, \boldsymbol{x})$ and therefore how Dirac spinors depend upon the momentum $p = (p^0, \boldsymbol{p})$. In particular, the combination

$$\psi(x) = (m - \gamma^a \partial_a)\, \mathrm{e}^{\pm \mathrm{i} p\cdot x} \chi = (m \mp \mathrm{i}\gamma^a p_a)\, \mathrm{e}^{\pm \mathrm{i} p\cdot x} \chi \tag{78}$$

for $p^2 = -m^2$ obeys the Dirac equation (65) for ***every*** constant four-component spinor χ [16]

$$(\gamma^b \partial_b + m)\psi(x) = (\pm \mathrm{i}\gamma^b p_b + m)(m \mp \mathrm{i}\gamma^a p_a)\, \mathrm{e}^{\pm \mathrm{i} p\cdot x} \chi = (m^2 + p^2)\, \mathrm{e}^{\pm \mathrm{i} p\cdot x} \chi = 0. \tag{79}$$

Since the u spinors in a Dirac field (91) occur with the phase $\mathrm{e}^{\mathrm{i}p\cdot x}$, we set $\chi = u(\mathbf{0}, s)$ and find that

$$\psi_{u,s}(x) = (m - \gamma^a \partial_a)\, u(\mathbf{0}, s)\, \mathrm{e}^{\mathrm{i}p\cdot x} = (m - \mathrm{i}\gamma^a p_a)\, u(\mathbf{0}, s)\, \mathrm{e}^{\mathrm{i}p\cdot x} \tag{80}$$

is a solution of the Dirac equation $(\gamma^b \partial_b + m)\psi_{u,s}(x) = 0$. So we set

$$u(\boldsymbol{p}, s) = \frac{m - \mathrm{i}\gamma^a p_a}{\sqrt{2p^0(p^0 + m)}}\, u(\mathbf{0}, s) \tag{81}$$

in which the square root normalizes the spinor.

Similarly since the v spinors in a Dirac field (91) occur with the phase $\mathrm{e}^{-\mathrm{i}p\cdot x}$, we set $\chi = v(\mathbf{0}, s)$ and find that

$$\psi_{v,s}(x) = (m - \gamma^a \partial_a)\, v(\mathbf{0}, s)\, \mathrm{e}^{-\mathrm{i}p\cdot x} \tag{82}$$

is a solution of the Dirac equation $(\gamma^b \partial_b + m)\psi_{v,s}(x) = 0$. So we set

$$v(\boldsymbol{p}, s) = \frac{m + \mathrm{i}\gamma^a p_a}{\sqrt{2p^0(p^0 + m)}}\, v(\mathbf{0}, s). \tag{83}$$

The vectors $u(\boldsymbol{p}, s)$ for $s = \pm\frac{1}{2}$ are two eigenvectors of $-\mathrm{i}\gamma^a p_a$ with eigenvalue m, and the vectors $v(\boldsymbol{p}, s)$ for $s = \pm\frac{1}{2}$ are two eigenvectors of $-\mathrm{i}\gamma^a p_a$ with eigenvalue $-m$.

In more detail, the spinors are [15]

$$u(\boldsymbol{p}, \tfrac{1}{2}) = \frac{1}{n(p^0)} \begin{pmatrix} m + p^0 - p_3 \\ -p_1 - \mathrm{i}p_2 \\ m + p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix}, \qquad u(\boldsymbol{p}, -\tfrac{1}{2}) = \frac{1}{n(p^0)} \begin{pmatrix} -p_1 + \mathrm{i}p_2 \\ m + p^0 + p_3 \\ p_1 - \mathrm{i}p_2 \\ m + p^0 - p_3 \end{pmatrix}$$
$$v(\boldsymbol{p}, \tfrac{1}{2}) = \frac{1}{n(p^0)} \begin{pmatrix} -p_1 + \mathrm{i}p_2 \\ m + p^0 + p_3 \\ -p_1 + \mathrm{i}p_2 \\ p_3 - m - p^0 \end{pmatrix}, \qquad v(\boldsymbol{p}, -\tfrac{1}{2}) = \frac{1}{n(p^0)} \begin{pmatrix} p_3 - m - p^0 \\ p_1 + \mathrm{i}p_2 \\ m + p^0 + p_3 \\ p_1 + \mathrm{i}p_2 \end{pmatrix} \tag{84}$$

in which the factor $n(p^0) = 2\sqrt{p^0(p^0+m)}$ ensures their normalization

$$u^\dagger(\boldsymbol{p},s)\,u(\boldsymbol{p},s') = \delta_{s,s'}, \qquad v^\dagger(\boldsymbol{p},s)\,v(\boldsymbol{p},s') = \delta_{s,s'}. \tag{85}$$

They obey the parity conditions (72)

$$u(\boldsymbol{p},s) = \mathrm{i}\gamma^0 u(-\boldsymbol{p},s) \quad \text{and} \quad v(\boldsymbol{p},s) = -\mathrm{i}\gamma^0 v(-\boldsymbol{p},s), \tag{86}$$

the charge-conjugation conditions (73)

$$u(\boldsymbol{p},s) = \gamma^2 v^*(\boldsymbol{p},s) \quad \text{and} \quad v(\boldsymbol{p},s) = \gamma^2 u^*(\boldsymbol{p},s), \tag{87}$$

and the time-reversal conditions (74)

$$u^*(\boldsymbol{p},s) = (-1)^{\frac{1}{2}+s}\gamma^1\gamma^3 u(-\boldsymbol{p},-s) \quad \text{and} \quad v^*(\boldsymbol{p},s) = (-1)^{\frac{1}{2}+s}\gamma^1\gamma^3 v(-\boldsymbol{p},-s). \tag{88}$$

If one switches to a different set of gamma matrices $\gamma'^i = S\,\gamma^i\,S^{-1}$, then one must also switch one's spinors to $u'(\boldsymbol{p},s) = S\,u(\boldsymbol{p},s)$ and $v'(\boldsymbol{p},s) = Sv(\boldsymbol{p},s)$.

8. Left-handed and right-handed spin-one-half fields

This Section describes how the upper two components and the lower two components of a Dirac field transform as left-handed and as right-handed fields under Lorentz transformations.

A 4-component Dirac field $\psi_D(x)$, $_{D=1,2,3,4,}$

$$\psi_D(x) = \sum_{s=\pm 1/2} \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \left[u_D(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\, a(\boldsymbol{p},s) + v_D(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\, a_\mathrm{c}^\dagger(\boldsymbol{p},s) \right] \tag{89}$$

is made of a left-handed 2-component field $\psi_\ell(x)$ and a right-handed 2-component field $\psi_r(x)$

$$\psi(x) = \begin{pmatrix} \psi_\ell(x) \\ \psi_r(x) \end{pmatrix} \tag{90}$$

and so has the form

$$\begin{pmatrix} \psi_\ell(x) \\ \psi_r(x) \end{pmatrix} = \sum_{s=\pm 1/2} \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \left[\begin{pmatrix} u_\ell(\boldsymbol{p},s) \\ u_r(\boldsymbol{p},s) \end{pmatrix} \mathrm{e}^{\mathrm{i}p\cdot x}\, a(\boldsymbol{p},s) + \begin{pmatrix} v_\ell(\boldsymbol{p},s) \\ v_r(\boldsymbol{p},s) \end{pmatrix} \mathrm{e}^{-\mathrm{i}p\cdot x}\, a_\mathrm{c}^\dagger(\boldsymbol{p},s) \right]. \tag{91}$$

We can get a better understanding of the left-handed and right-handed spinors and fields by rewriting the spinor formulas (81) and (83). To do that, we recall that the zero-momentum spinors (71) obey the parity conditions (72)

$$u(\boldsymbol{0},s) = \mathrm{i}\gamma^0 u(\boldsymbol{0},s) \quad \text{and} \quad v(\boldsymbol{0},s) = -\mathrm{i}\gamma^0 v(\boldsymbol{0},s). \tag{92}$$

These conditions let us replace $-\mathrm{i}\gamma^a p_a u(\boldsymbol{0},s)$ by $\gamma^a\gamma^0 p_a u(\boldsymbol{0},s)$ and $\mathrm{i}\gamma^a p_a v(\boldsymbol{0},s)$ by $\gamma^a\gamma^0 p_a v(\boldsymbol{0},s)$ without disturbing the definitions (81) and (83) of the 4-spinors u and v

$$u(\boldsymbol{p}, s) = \frac{m + \gamma^a \gamma^0 p_a}{\sqrt{2p^0(p^0+m)}}\, u(\boldsymbol{0}, s) \quad \text{and} \quad v(\boldsymbol{p}, s) = \frac{m + \gamma^a \gamma^0 p_a}{\sqrt{2p^0(p^0+m)}}\, v(\boldsymbol{0}, s). \tag{93}$$

We now see that the 4-spinors $u(\boldsymbol{p}, s)$ and $v(\boldsymbol{p}, s)$ are generated from the zero-momentum spinors $u(\boldsymbol{0}, s)$ and $v(\boldsymbol{0}, s)$ by the same matrix $m + \gamma^a \gamma^0 p_a$

$$m + \gamma^a \gamma^0 p_a = \begin{pmatrix} m + p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & m + p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{pmatrix} \tag{94}$$

which is block diagonal.

The upper-left block $m + p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma}$ is proportional to the left-handed 2×2 representation $D^{(1/2,0)}(L(p))$ of the Lorentz transformation $L(p)$ that takes momentum $(m, \boldsymbol{0})$ to $p = (p^0, \boldsymbol{p})$ via a boost in the $\hat{\boldsymbol{p}}$ direction [12, 15]. The matrix $D^{(1/2,0)}(L(p))$ is

$$D^{(1/2,0)}(L(p)) = \frac{m + p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}} = \exp\left(-\,\alpha\, \hat{\boldsymbol{p}}\cdot\frac{\boldsymbol{\sigma}}{2}\right) \tag{95}$$

in which $\alpha = \operatorname{arctanh}(|\boldsymbol{p}|/p^0)$ [13, 15].

The lower-right block $m + p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma}$ is proportional to the right-handed 2×2 representation $D^{(0,1/2)}(L(p))$ of the same Lorentz transformation $L(p)$

$$D^{(0,1/2)}(L(p)) = \frac{m + p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}} = \exp\left(\alpha\, \hat{\boldsymbol{p}}\cdot\frac{\boldsymbol{\sigma}}{2}\right). \tag{96}$$

If we combine the left- and right-handed representations of $L(p)$ into a single 4×4 matrix

$$D^{(1/2,0)\oplus(0,1/2)}(L(p)) = \begin{pmatrix} D^{(1/2,0)}(L(p)) & 0 \\ 0 & D^{(0,1/2)}(L(p)) \end{pmatrix}, \tag{97}$$

then we can write the formulas (93) for the spinors as

$$\begin{aligned} u(\boldsymbol{p}, s) &= \sqrt{\frac{m}{p^0}}\, D^{(1/2,0)\oplus(0,1/2)}(L(p))\, u(\boldsymbol{0}, s) \\ v(\boldsymbol{p}, s) &= \sqrt{\frac{m}{p^0}}\, D^{(1/2,0)\oplus(0,1/2)}(L(p))\, v(\boldsymbol{0}, s) \end{aligned} \tag{98}$$

or equivalently as

$$\begin{aligned} u(\boldsymbol{p}, s) &= \begin{pmatrix} u_\ell(\boldsymbol{p}, s) \\ u_r(\boldsymbol{p}, s) \end{pmatrix} = \sqrt{\frac{m}{p^0}} \begin{pmatrix} D^{(1/2,0)}(L(p))\, u_\ell(\boldsymbol{0}, s) \\ D^{(0,1/2)}(L(p))\, u_r(\boldsymbol{0}, s) \end{pmatrix} \\ v(\boldsymbol{p}, s) &= \begin{pmatrix} v_\ell(\boldsymbol{p}, s) \\ v_r(\boldsymbol{p}, s) \end{pmatrix} = \sqrt{\frac{m}{p^0}} \begin{pmatrix} D^{(1/2,0)}(L(p))\, v_\ell(\boldsymbol{0}, s) \\ D^{(0,1/2)}(L(p))\, v_r(\boldsymbol{0}, s) \end{pmatrix}. \end{aligned} \tag{99}$$

Weinberg [17] has shown that a Dirac field (91) transforms under a Lorentz transformation Λ as

$$U(\Lambda)\, \psi_D(x)\, U^{-1}(\Lambda) = D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda^{-1}) \sum_{D'=1}^{4} \psi_{D'}(\Lambda x) \tag{100}$$

if and only if the spinors are defined by equations (71) and (98). A derivation of this result is sketched in appendix B.

Since the 4×4 matrix $D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda^{-1})$ is block diagonal, we can separate equation (100), that says how a Dirac field transforms under a Lorentz transformation Λ, into one equation for the left-handed field ψ_ℓ and another for the right-handed field ψ_r

$$\begin{aligned} U(\Lambda)\psi_\ell(x)U^{-1}(\Lambda) &= D^{(1/2,0)}(\Lambda^{-1})\,\psi_\ell(\Lambda x) \\ U(\Lambda)\psi_r(x)U^{-1}(\Lambda) &= D^{(0,1/2)}(\Lambda^{-1})\,\psi_r(\Lambda x). \end{aligned} \tag{101}$$

To see why u_ℓ and v_ℓ are called left handed and why u_r and v_r are called right handed, we look at the terms $u_\ell(\boldsymbol{p},s)\mathrm{e}^{\mathrm{i}p\cdot x}a(\boldsymbol{p},s)$, $v_\ell(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^\dagger_\mathrm{c}(\boldsymbol{p},s)$, $u_r(\boldsymbol{p},s)\mathrm{e}^{\mathrm{i}p\cdot x}a(\boldsymbol{p},s)$, and $v_r(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^\dagger_\mathrm{c}(\boldsymbol{p},s)$ in the Dirac field (91) for particles with momentum in the z direction $\boldsymbol{p} = p\hat{\boldsymbol{z}}$ in the limit $m/p^0 \to 0$, a limit reached by neutrinos with $p^0 \gtrsim 1$ keV. In the $m/p^0 \to 0$ limit, the spinors (84) for $\boldsymbol{p} = p\hat{\boldsymbol{z}}$ are

$$\begin{aligned} &\begin{pmatrix} u_\ell(p\hat{\boldsymbol{z}},\frac{1}{2}) \\ u_r(p\hat{\boldsymbol{z}},\frac{1}{2}) \end{pmatrix} \to \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \qquad \begin{pmatrix} u_\ell(p\hat{\boldsymbol{z}},-\frac{1}{2}) \\ u_r(p\hat{\boldsymbol{z}},-\frac{1}{2}) \end{pmatrix} \to \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \\ &\begin{pmatrix} v_\ell(p\hat{\boldsymbol{z}},\frac{1}{2}) \\ v_r(p\hat{\boldsymbol{z}},\frac{1}{2}) \end{pmatrix} \to \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} v_\ell(p\hat{\boldsymbol{z}},-\frac{1}{2}) \\ v_r(p\hat{\boldsymbol{z}},-\frac{1}{2}) \end{pmatrix} \to \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} \end{aligned} \tag{102}$$

and so the nonzero terms $u_D(\boldsymbol{p},s)\mathrm{e}^{\mathrm{i}p\cdot x}a(\boldsymbol{p},s)$ and $v_D(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^\dagger_\mathrm{c}(\boldsymbol{p},s)$ are

$$\begin{aligned} &u_r(p\hat{\boldsymbol{z}},\tfrac{1}{2})\,\mathrm{e}^{\mathrm{i}p(x)}\,a(p\hat{\boldsymbol{z}},\tfrac{1}{2}), \qquad u_\ell(p\hat{\boldsymbol{z}},-\tfrac{1}{2})\,\mathrm{e}^{\mathrm{i}p(z-t)}\,a(p\hat{\boldsymbol{z}},-\tfrac{1}{2}), \\ &v_\ell(p\hat{\boldsymbol{z}},\tfrac{1}{2})\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^\dagger_\mathrm{c}(p\hat{\boldsymbol{z}},\tfrac{1}{2}), \quad \text{and} \quad v_r(p\hat{\boldsymbol{z}},-\tfrac{1}{2})\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^\dagger_\mathrm{c}(p\hat{\boldsymbol{z}},-\tfrac{1}{2}). \end{aligned} \tag{103}$$

The field $\ell(x)$ is said to be left handed because it contains $u_\ell(p\hat{\boldsymbol{z}},-\frac{1}{2})\,a(p\hat{\boldsymbol{z}},-\frac{1}{2})$ which destroys particles with momentum antiparallel to the spin and $v_\ell(p\hat{\boldsymbol{z}},\frac{1}{2})\,a^\dagger_\mathrm{c}(p\hat{\boldsymbol{z}},\frac{1}{2})$ which creates antiparticles with momentum parallel to the spin. The field $r(x)$ is said to be right handed because it contains $u_r(p\hat{\boldsymbol{z}},\frac{1}{2})\,a(p\hat{\boldsymbol{z}},\frac{1}{2})$ which destroys particles with momentum parallel to the spin and $v_r(p\hat{\boldsymbol{z}},-\frac{1}{2})\,a^\dagger_\mathrm{c}(p\hat{\boldsymbol{z}},-\frac{1}{2})$ which creates antiparticles with momentum antiparallel to the spin. The weak gauge group $SU(2)_\ell$ acts on left-handed fields.

The simplest spin-one-half fields are the 2-component Majorana fields that are linear combinations of the annihilation operators $a(\boldsymbol{p},s)$ and their adjoints $a^\dagger(\boldsymbol{p},s)$ multiplied by left-handed and right-handed 2-component spinors (99)

$$\begin{aligned} \psi_{\ell M}(x) &= \sum_{s=\pm 1/2} \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u_\ell(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,a(\boldsymbol{p},s) + v_\ell(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^\dagger(\boldsymbol{p},s) \\ \psi_{rM}(x) &= \sum_{s=\pm 1/2} \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u_r(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,a(\boldsymbol{p},s) + v_r(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^\dagger(\boldsymbol{p},s). \end{aligned} \tag{104}$$

Together they make a 4-component Majorana field

$$\psi_M(x) = \begin{pmatrix} \psi_{\ell M}(x) \\ \psi_{rM}(x) \end{pmatrix}. \tag{105}$$

If one has two annihilation operators $a_1(\boldsymbol{p}, s)$ and $a_2(\boldsymbol{p}, s)$ and their adjoints $a_1^\dagger(\boldsymbol{p}, s)$ and $a_2^\dagger(\boldsymbol{p}, s)$, all referring to particles of the same mass, then one may make operators that annihilation and create particles and their antiparticles

$$\begin{aligned} a(\boldsymbol{p}, s) &= \frac{1}{\sqrt{2}}\left[a_1(\boldsymbol{p}, s) + \mathrm{i}a_2(\boldsymbol{p}, s)\right], & a_c(\boldsymbol{p}, s) &= \frac{1}{\sqrt{2}}\left[a_1(\boldsymbol{p}, s) - \mathrm{i}a_2(\boldsymbol{p}, s)\right] \\ a^\dagger(\boldsymbol{p}, s) &= \frac{1}{\sqrt{2}}\left[a_1^\dagger\, \boldsymbol{p}, s) - \mathrm{i}a_2^\dagger(\boldsymbol{p}, s)\right], & a_c^\dagger(\boldsymbol{p}, s) &= \frac{1}{\sqrt{2}}\left[a_1^\dagger(\boldsymbol{p}, s) + \mathrm{i}a_2^\dagger(\boldsymbol{p}, s)\right] \end{aligned} \tag{106}$$

and define a Dirac field as

$$\psi(x) = \frac{1}{\sqrt{2}}\left[\psi_{M1}(x) + \mathrm{i}\psi_{M1}(x)\right] = \begin{pmatrix} \psi_\ell(x) \\ \psi_r(x) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_{\ell M1}(x) + \mathrm{i}\psi_{\ell M2}(x) \\ \psi_{rM1}(x) + \mathrm{i}\psi_{rM1}(x) \end{pmatrix}. \tag{107}$$

9. Conclusions

The Dirac spinors at momentum zero $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s)$ cannot be chosen arbitrarily as four orthonormal 4-component vectors. They are instead determined (71) by the requirement that the Dirac field transform correctly under rotations and obey the Dirac equation as explained in sections 3–6.

Once one has the zero-momentum spinors, the Dirac equation yields the finite-momentum spinors (81) and (83) as explained in section 7.

A 4-component Dirac field ψ is composed of a 2-component left-handed field ψ_ℓ and a 2-component right-handed field ψ_r as described in section 8.

Appendix A shows that when spinors are defined correctly (71), (81), and (83), a Dirac field transforms appropriately under charge conjugation, parity, and time reversal. Appendix B explains how particles and fields transform under Lorentz transformations and shows that the spinors of a Dirac field that transforms correctly under Lorentz transformations has spinors that are related to its zero-momentum spinors by equations (81) and (83). Appendix C explains that a Dirac field is a complex linear combination of two Majorana fields of the same mass.

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Appendix A. Charge conjugation, parity, and time reversal

Because spinors obey the parity, charge-conjugation, and time-reversal conditions (86)–(88), Dirac fields transform simply under parity, charge conjugation, and time reversal.

Parity reverses space and momentum, so the annihilation and creation operators obey the rules [18]

$$\mathsf{P}\, a_{\mathrm{c}}^{\dagger}(\boldsymbol{p},s)\,\mathsf{P}^{-1} = \eta_{\mathrm{c}}\, a_{\mathrm{c}}^{\dagger}(-\boldsymbol{p},s) = -\eta^{*}\, a_{\mathrm{c}}^{\dagger}(-\boldsymbol{p},s) \quad \text{and} \quad \mathsf{P}\, a(\boldsymbol{p},s)\,\mathsf{P}^{-1} = \eta^{*}\, a(-\boldsymbol{p},s). \tag{A1}$$

They and the parity conditions (86) imply that a Dirac field transforms simply under parity

$$\begin{aligned}
\mathsf{P}\,\psi(t,\boldsymbol{x})\,\mathsf{P}^{-1} &= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,\mathsf{P}a(\boldsymbol{p},s)\mathsf{P}^{-1} + v(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,\mathsf{P}a_{\mathrm{c}}^{\dagger}(\boldsymbol{p},s)\mathsf{P}^{-1}\\
&= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,\eta^{*}a(-\boldsymbol{p},s) + v(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,\eta_{\mathrm{c}}\,a_{\mathrm{c}}^{\dagger}(-\boldsymbol{p},s)\\
&= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u(-\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot Px}\,\eta^{*}a(\boldsymbol{p},s) + v(-\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot Px}\,\eta_{\mathrm{c}}\,a_{\mathrm{c}}^{\dagger}(\boldsymbol{p},s)\\
&= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; \mathrm{i}\gamma^0 u(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot Px}\,\eta^{*}a(\boldsymbol{p},s) - \mathrm{i}\gamma^0 v(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot Px}\,\eta_{\mathrm{c}}\,a_{\mathrm{c}}^{\dagger}(\boldsymbol{p},s)\\
&= \eta^{*}\mathrm{i}\gamma^0 \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot Px}\,a(\boldsymbol{p},s) + v(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot Px}\,a_{\mathrm{c}}^{\dagger}(\boldsymbol{p},s)\\
&= \eta^{*}\mathrm{i}\gamma^0\,\psi(t,-\boldsymbol{x}).
\end{aligned} \tag{A2}$$

A Majorana field (105) creates and destroys the same kind of particle. Its Fourier expansion is that of a Dirac field but with $a_{\mathrm{c}}^{\dagger} = a^{\dagger}$

$$\psi_M(x) = \sum_{s=\pm 1/2} \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,a(\boldsymbol{p},s) + v(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^{\dagger}(\boldsymbol{p},s). \tag{A3}$$

For a Majorana particle, $a_{\mathrm{c}}^{\dagger}(\boldsymbol{p},s) = a^{\dagger}(\boldsymbol{p},s)$, so $\eta_{\mathrm{c}} = \eta$. But the parity rules (A1) also require that $\eta_{\mathrm{c}} = -\eta^{*}$, so η must be imaginary, $\eta = -\eta^{*}$.

In general, charge conjugation maps particles into their antiparticles [19]

$$\mathsf{C}\, a_{\mathrm{c}}^{\dagger}(p,s)\,\mathsf{C}^{-1} = \xi_{\mathrm{c}} a^{\dagger}(p,s) = \xi^{*} a^{\dagger}(p,s) \quad \text{and} \quad \mathsf{C}\, a(p,s)\,\mathsf{C}^{-1} = \xi^{*} a_{\mathrm{c}}(p,s). \tag{A4}$$

The charge-conjugation conditions (87) and the definition (A4) of charge conjugation imply that a Dirac field (89) transforms simply under charge conjugation

$$\begin{aligned}
\mathsf{C}\,\psi(x)\,\mathsf{C}^{-1} &= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,\mathsf{C}a(\boldsymbol{p},s)\mathsf{C}^{-1} + v(\boldsymbol{p},s)\mathrm{e}^{-\mathrm{i}p\cdot x}\,\mathsf{C}a_{\mathrm{c}}^{\dagger}(\boldsymbol{p},s)\mathsf{C}^{-1}\\
&= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; u(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,\xi^{*}a_{\mathrm{c}}(\boldsymbol{p},s) + v(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,\xi_{\mathrm{c}}a^{\dagger}(\boldsymbol{p},s)\\
&= \xi^{*}\sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\; \gamma^2 v^{*}(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,a_{\mathrm{c}}(\boldsymbol{p},s) + \gamma^2 u^{*}(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a^{\dagger}(\boldsymbol{p},s)
\end{aligned}$$

$$= \xi^* \gamma^2 \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}} \; u^*(\boldsymbol{p}, s)\, e^{-ip\cdot x}\, a^\dagger(\boldsymbol{p}) + v^*(\boldsymbol{p}, s)\, e^{ip\cdot x}\, a_c(\boldsymbol{p}, s)$$
$$= \xi^* \gamma^2 \, \psi^*(x) \tag{A5}$$

in which the asterisk denotes complex and Hermitian conjugation but not the conversion of column vectors into row vectors.

For Majorana particles, the charge-conjugation conditions (87) imply that $\xi_c a^\dagger(p,s) = \mathsf{C}\, a_c^\dagger(p,s)\, \mathsf{C}^{-1} = \mathsf{C} a^\dagger(p,s)\mathsf{C}^{-1} = \xi a^\dagger(p,s)$, so the phase $\xi_c = \xi^* = \xi$ is real for Majorana particles. The charge-conjugation conditions (87) also imply that Hermitian conjugation changes a Majorana field (A3) to

$$\begin{aligned} \psi_M^* &= \sum_{s=\pm 1/2} \int \frac{d^3 p}{(2\pi)^{3/2}} \; u^*(\boldsymbol{p}, s)\, e^{-ip\cdot x}\, a^\dagger(\boldsymbol{p}, s) + v^*(\boldsymbol{p}, s)\, e^{ip\cdot x}\, a(\boldsymbol{p}, s) \\ &= \sum_{s=\pm 1/2} \int \frac{d^3 p}{(2\pi)^{3/2}} \; \gamma^2 u(\boldsymbol{p}, s)\, e^{ip\cdot x}\, a(\boldsymbol{p}, s) + \gamma^2 v(\boldsymbol{p}, s)\, e^{-ip\cdot x}\, a^\dagger(\boldsymbol{p}, s) \end{aligned} \tag{A6}$$

so that a Majorana field obeys the Majorana condition

$$\psi_M^*(x) = \gamma^2 \, \psi_M(x). \tag{A7}$$

Time reversal reverses momentum and spin, adds a phase, and complex conjugates complex numbers [20]

$$\begin{aligned} \mathsf{T}\, z\, a(\boldsymbol{p}, s)\, \mathsf{T}^{-1} &= z^*\, (-1)^{\frac{1}{2}-s}\, \zeta^*\, a(-\boldsymbol{p}, -s) \\ \mathsf{T}\, w\, a_c^\dagger(\boldsymbol{p}, s)\, \mathsf{T}^{-1} &= w^*\, (-1)^{\frac{1}{2}-s}\, \zeta_c\, a_c^\dagger(-\boldsymbol{p}, -s) \end{aligned} \tag{A8}$$

in which z and w are arbitrary complex numbers, and $\zeta_c = \zeta^*$. This definition (A8) and the time-reversal conditions (88) imply that a Dirac field (89) transforms simply under time reversal

$$\begin{aligned} \mathsf{T}\, \psi(t, \boldsymbol{x})\, \mathsf{T}^{-1} &= \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}} \; u^*(\boldsymbol{p}, s)\, e^{-ip\cdot x}\, \mathsf{T} a(\boldsymbol{p}, s) \mathsf{T}^{-1} + v^*(\boldsymbol{p}, s)\, e^{ip\cdot x}\, \mathsf{T} a_c^\dagger(\boldsymbol{p}, s) \mathsf{T}^{-1} \\ &= \sum_s (-1)^{\frac{1}{2}-s} \int \frac{d^3 p}{(2\pi)^{3/2}}\, u^*(\boldsymbol{p}, s)\, e^{-ip\cdot x} \zeta^*\, a(-\boldsymbol{p}, -s) + v^*(\boldsymbol{p}, s)\, e^{ip\cdot x} \zeta_c\, a_c^\dagger(-\boldsymbol{p}, -s) \\ &= \sum_s (-1)^{\frac{1}{2}-s} \int \frac{d^3 p}{(2\pi)^{3/2}}\, u^*(\boldsymbol{p}, s)\, e^{-ip\cdot x} \zeta^*\, a(-\boldsymbol{p}, -s) + v^*(\boldsymbol{p}, s)\, e^{ip\cdot x} \zeta^*\, a_c^\dagger(-\boldsymbol{p}, -s) \\ &= \zeta^* \sum_s (-1)^{\frac{1}{2}-s} \int \frac{d^3 p}{(2\pi)^{3/2}}\, u^*(\boldsymbol{p}, s)\, e^{-ip\cdot x} a(-\boldsymbol{p}, -s) + v^*(\boldsymbol{p}, s)\, e^{ip\cdot x} a_c^\dagger(-\boldsymbol{p}, -s) \\ &= \zeta^* \gamma^1 \gamma^3 \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}}\, u(-\boldsymbol{p}, -s)\, e^{-ip\cdot x} a(-\boldsymbol{p}, -s) + v(-\boldsymbol{p}, -s)\, e^{ip\cdot x} a_c^\dagger(-\boldsymbol{p}, -s) \\ &= \zeta^* \gamma^1 \gamma^3 \, \psi(-t, \boldsymbol{x}). \end{aligned} \tag{A9}$$

Appendix B. Wigner rotations

Steven Weinberg has shown that the Lorentz-transformation properties of a quantum field of any spin determine the spinors of that field [21]. This appendix repeats that derivation for fields of spin one-half and inserts some extra steps that may help students.

In section 3 we considered how rotations change states $|\mathbf{0}, s\rangle$ without saying how Lorentz boosts change them to states of finite momentum. Because we will be talking about Lorentz transformations, it will be convenient to write states as $|(p^0, \boldsymbol{p}), s\rangle \equiv |p, s\rangle$. The state $|p, s\rangle$ is the image of the state of a particle at rest $|0, s\rangle$ under the unitary transformation $U(L(p))$ that implements the standard Lorentz boost $L(p)$ in the direction $\boldsymbol{p}$

$$|p, s\rangle = \sqrt{\frac{m}{p^0}}\, U(L(p))\, |0, s\rangle \tag{B1}$$

normalized so that $\langle p, s|p', s'\rangle = \delta_{ss'}\delta(\boldsymbol{p} - \boldsymbol{p}')$. A Lorentz transformation Λ changes the state $|p, s\rangle$ to a linear combination of states with momentum Λp but with different spins in the z direction

$$U(\Lambda)|p, s\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'=\pm 1/2} D_{s's}(R_W(\Lambda, p))\, |\Lambda p, s'\rangle. \tag{B2}$$

The 2×2 matrix $D(R_W(\Lambda, p))$ as in equation (15) represents the Wigner rotation [22, 23]

$$R_W(\Lambda, p) = L^{-1}(\Lambda p)\, \Lambda\, L(p) \tag{B3}$$

that boosts a particle at rest to momentum p, and then to Λp, and then back to rest. The Wigner rotation arises because

$$\begin{aligned} U(\Lambda)|p, s\rangle &= N(p)\, U(\Lambda p)\, U(L^{-1}(\Lambda p))\, U(\Lambda)\, U(L(p))\, |0, s\rangle \\ &= N(p)\, U(\Lambda p)\, U(R_W(\Lambda, p))\, |0, s\rangle \\ &= N(p)\, U(\Lambda p) \sum_{s'=-1/2}^{1/2} D_{s's}(R_W(\Lambda, p))\, |0, s'\rangle \\ &= \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'=-1/2}^{1/2} D_{s's}(R_W(\Lambda, p))\, |\Lambda p, s'\rangle. \end{aligned} \tag{B4}$$

So the generalizations to Lorentz transformations of states of finite momentum of the formulas (30) for how zero-momentum creation and annihilation operators transform under rotations are

$$\begin{aligned} U(\Lambda)\, a^\dagger(p, s)\, U^{-1}(\Lambda) &= \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'=-1/2}^{1/2} D^*_{ss'}(R_W^{-1}(\Lambda, p))\, a^\dagger(\Lambda p, s) \\ U(\Lambda)\, a(p, s)\, U^{-1}(\Lambda) &= \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'=-1/2}^{1/2} D_{ss'}(R_W^{-1}(\Lambda, p))\, a(\Lambda p, s'). \end{aligned} \tag{B5}$$

Thus a Dirac field (89) will transform correctly under a Lorentz transformation Λ (100)

$$
\begin{aligned}
U(\Lambda)\,\psi_D(x)\,U^{-1}(\Lambda) = {} & \sum_{s=\pm 1/2} \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \left[u_D(p,s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\, U(\Lambda) a(p,s) U^{-1}(\Lambda) \right. \\
& \left. + v_D(p,s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\, U(\Lambda) a_{\mathrm{c}}^{\dagger}(p,s) U^{-1}(\Lambda) \right] \\
= {} & \sum_{D'=1}^{4} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda^{-1}) \psi_{D'}(\Lambda x) \\
= {} & \sum_{D'=1}^{4} \sum_{s=\pm 1/2} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda^{-1}) \\
& \times \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \left[u_{D'}(p,s)\,\mathrm{e}^{\mathrm{i}p\cdot\Lambda x}\, a(p,s) \right. \\
& \left. + v_{D'}(p,s)\,\mathrm{e}^{-\mathrm{i}p\cdot\Lambda x}\, a_{\mathrm{c}}^{\dagger}(p,s) \right]
\end{aligned} \tag{B6}
$$

if

$$
\begin{aligned}
& \sum_{s,s'=\pm 1/2} \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{(\Lambda p)^0}{p^0}}\, u_D(p,s)\,\mathrm{e}^{\mathrm{i}p\cdot x} D_{ss'}(R_W^{-1}(\Lambda,p))\, a(\Lambda p, s') \\
& \quad = \sum_{D'=1}^{4} \sum_{s=\pm 1/2} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda^{-1}) \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} u_{D'}(p,s)\,\mathrm{e}^{\mathrm{i}p\cdot\Lambda x}\, a(p,s)
\end{aligned} \tag{B7}
$$

and

$$
\begin{aligned}
& \sum_{s,s'=\pm 1/2} \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \sqrt{\frac{(\Lambda p)^0}{p^0}}\, v_D(p,s)\,\mathrm{e}^{-\mathrm{i}p\cdot x} D^{*}_{ss'}(R_W^{-1}(\Lambda,p))\, a_{\mathrm{c}}^{\dagger}(\Lambda p, s') \\
& \quad = \sum_{D'=1}^{4} \sum_{s=\pm 1/2} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}\left(\Lambda^{-1}\right) \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} v_{D'}(p,s) \mathrm{e}^{-\mathrm{i}p\cdot\Lambda x}\, a_{\mathrm{c}}^{\dagger}(p,s).
\end{aligned} \tag{B8}
$$

Setting $\mathrm{d}^3 p = p^0 \mathrm{d}^3 \Lambda p/(\Lambda p)^0)$ in the left-hand sides of these equations (B7) and (B8), and then on their right-hand sides changing variables $p \to \Lambda p$ and using $\Lambda p \cdot \Lambda x = p \cdot x$, we get

$$
\begin{aligned}
& \sum_{s,s'=\pm 1/2} \int \frac{\mathrm{d}^3 \Lambda p}{(2\pi)^{3/2}} \sqrt{\frac{p^0}{(\Lambda p)^0}}\, u_D(p,s)\,\mathrm{e}^{\mathrm{i}p\cdot x} D_{ss'}(R_W^{-1}(\Lambda,p))\, a(\Lambda p, s') \\
& \quad = \sum_{D'=1}^{4} \sum_{s=\pm 1/2} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda^{-1}) \int \frac{\mathrm{d}^3 \Lambda p}{(2\pi)^{3/2}}\, u_{D'}(\Lambda p, s) \mathrm{e}^{\mathrm{i}p\cdot x}\, a(\Lambda p, s)
\end{aligned} \tag{B9}
$$

and

$$\sum_{s,s'=\pm 1/2} \int \frac{\mathrm{d}^3\Lambda p}{(2\pi)^{3/2}} \sqrt{\frac{p^0}{(\Lambda p)^0}}\, v_D(p,s)\, \mathrm{e}^{-\mathrm{i}p\cdot x} D^*_{ss'}(R_W^{-1}(\Lambda,p))\, a^\dagger_{\mathrm{c}}(\Lambda p, s')$$
$$= \sum_{D'=1}^{4} \sum_{s=\pm 1/2} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda^{-1}) \int \frac{\mathrm{d}^3\Lambda p}{(2\pi)^{3/2}}\, v_{D'}(\Lambda p, s) \mathrm{e}^{-\mathrm{i}p\cdot x}\, a^\dagger_{\mathrm{c}}(\Lambda p, s). \tag{B10}$$

By equating the coefficients of $\mathrm{e}^{\mathrm{i}p\cdot x} a(\Lambda p, s)$ in (B9) and of $\mathrm{e}^{-\mathrm{i}p\cdot x} a^\dagger_{\mathrm{c}}(\Lambda p, s)$ in (B10), we find

$$\sum_{s'} \sqrt{\frac{p^0}{(\Lambda p)^0}} u_D(p,s')\, D^{-1}_{s's}(R_W(\Lambda,p)) = \sum_{D'} D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, u_{D'}(\Lambda p, s)$$
$$\sum_{s'} \sqrt{\frac{p^0}{(\Lambda p)^0}}\, v_D(p,s')\, D^{-1^*}_{s's}(R_W(\Lambda,p)) = \sum_{D'=1}^{4} D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, v_{D'}(\Lambda p, s). \tag{B11}$$

We now multiply these equations by the matrices $D_{ss''}(R_W(\Lambda,p))$ and $D^*_{ss''}(R_W(\Lambda,p))$ and sum over s

$$\sum_{s,s'} \sqrt{\frac{p^0}{(\Lambda p)^0}} u_D(p,s')\, D^{-1}_{s's}(R_W(\Lambda,p)) D_{ss''}(R_W(\Lambda,p))$$
$$= \sum_{s,D'} D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, u_{D'}(\Lambda p, s) D_{ss''}(R_W(\Lambda,p))$$
$$\sum_{s,s'} \sqrt{\frac{p^0}{(\Lambda p)^0}}\, v_D(p,s')\, D^{-1^*}_{s's}(R_W(\Lambda,p)) D^*_{ss''}(R_W(\Lambda,p))$$
$$= \sum_{s,D'} D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, v_{D'}(\Lambda p, s) D^*_{ss''}(R_W(\Lambda,p)). \tag{B12}$$

We then get

$$\sqrt{\frac{p^0}{(\Lambda p)^0}} u_D(p,s'') = \sum_{s,D'} D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, u_{D'}(\Lambda p, s) D_{ss''}(R_W(\Lambda,p))$$
$$\sqrt{\frac{p^0}{(\Lambda p)^0}}\, v_D(p,s'') = \sum_{s,D'} D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, v_{D'}(\Lambda p, s) D^*_{ss''}(R_W(\Lambda,p)). \tag{B13}$$

We now multiply equations (B13) by the matrix $D^{(1/2,0)\oplus(0,1/2)}_{D'',D}(\Lambda)$ and sum over D and D'

$$\sum_{D} \sqrt{\frac{p^0}{(\Lambda p)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D'',D}(\Lambda)\, u_D(p,s'')$$
$$= \sum_{s,D,D'} D^{(1/2,0)\oplus(0,1/2)}_{D'',D}(\Lambda) D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, u_{D'}(\Lambda p, s) D_{ss''}(R_W(\Lambda,p))$$

$$\sum_D \sqrt{\frac{p^0}{(\Lambda p)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D'',D}(\Lambda)\, v_D(p,s''),$$
$$= \sum_{s,D,D'} D^{(1/2,0)\oplus(0,1/2)}_{D'',D}(\Lambda) D^{(1/2,0)\oplus(0,1/2)-1}_{D,D'}(\Lambda)\, v_{D'}(\Lambda p,s) D^*_{ss''}(R_W(\Lambda,p)). \tag{B14}$$

These equations are more simply

$$\begin{aligned}\sum_D \sqrt{\frac{p^0}{(\Lambda p)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D'',D}(\Lambda)\, u_D(p,s'') &= \sum_s u_{D''}(\Lambda p,s) D_{ss''}(R_W(\Lambda,p))\\ \sum_D \sqrt{\frac{p^0}{(\Lambda p)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D'',D}(\Lambda)\, v_D(p,s'') &= \sum_s v_{D''}(\Lambda p,s) D^*_{ss''}(R_W(\Lambda,p)).\end{aligned} \tag{B15}$$

We set $\boldsymbol{p} = 0$ and $p(0) = (m, \boldsymbol{0})$, drop some primes, and interchange the right- and left-hand sides of these equations

$$\begin{aligned}u_D(\Lambda p(0),s) &= \sum_{D'} \sqrt{\frac{m}{(\Lambda p_0)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda)\, u_{D'}(p(0),s)\\ v_D(\Lambda p(0),s) &= \sum_{D'} \sqrt{\frac{m}{(\Lambda p)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda)\, v_{D'}(p(0),s).\end{aligned} \tag{B16}$$

Setting $\Lambda = L(p)$, the standard boost that takes $p(0)$ to p,

$$\begin{aligned}u_D(p,s) &= \sum_{D'} \sqrt{\frac{m}{(\Lambda p_0)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda)\, u_{D'}(p(0),s)\\ v_D(p,s) &= \sum_{D'} \sqrt{\frac{m}{(\Lambda p)^0}} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda)\, v_{D'}(p(0),s)\end{aligned} \tag{B17}$$

and switching back to using 3-momenta to label spinors, we find

$$\begin{aligned}u_D(\boldsymbol{p},s) &= \sum_{D'} \sqrt{\frac{m}{p^0}} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda)\, u_{D'}(0,s)\\ v_D(\boldsymbol{p},s) &= \sum_{D'} \sqrt{\frac{m}{p^0}} D^{(1/2,0)\oplus(0,1/2)}_{D,D'}(\Lambda)\, v_{D'}(0,s)\end{aligned} \tag{B18}$$

which are the desired formulas (98) that express the spinors at finite momentum in terms of the spinors at zero momentum.

Appendix C. Majorana and Dirac fields

We have seen (104)–(107) that a Dirac field (89) is a complex linear combination of two Majorana fields (A3) of the same mass

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{2}}\left[\psi_{M1}(x) + \mathrm{i}\psi_{M2}(x)\right] \\ &= \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \left\{u(\boldsymbol{p},s)\mathrm{e}^{\mathrm{i}p\cdot x}\left[a_1(\boldsymbol{p},s) + \mathrm{i}a_2(\boldsymbol{p},s)\right] + v(\boldsymbol{p},s)\right. \\ &\quad \left.\times\, \mathrm{e}^{-\mathrm{i}p\cdot x}\left[a_1^\dagger(\boldsymbol{p},s) + \mathrm{i}a_2^\dagger(\boldsymbol{p},s)\right]\right\} \\ &= \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \left[u_D(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,a(\boldsymbol{p},s) + v_D(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,a_{\mathrm{c}}^\dagger(\boldsymbol{p},s)\right] \end{aligned} \tag{C1}$$

and that the annihilation and creation operators of the Dirac field are complex linear combinations of the annihilation and creation operators of two Majorana fields ψ_{M1} and ψ_{M2}

$$\begin{aligned} a(\boldsymbol{p},s) &= \frac{1}{\sqrt{2}}\left[a_1(\boldsymbol{p},s) + \mathrm{i}a_2(\boldsymbol{p},s)\right], \qquad a_{\mathrm{c}}(\boldsymbol{p},s) = \frac{1}{\sqrt{2}}\left[a_1(\boldsymbol{p},s) - \mathrm{i}a_2(\boldsymbol{p},s)\right] \\ a^\dagger(\boldsymbol{p},s) &= \frac{1}{\sqrt{2}}\left[a_1^\dagger\,\boldsymbol{p},s) - \mathrm{i}a_2^\dagger(\boldsymbol{p},s)\right], \qquad a_{\mathrm{c}}^\dagger(\boldsymbol{p},s) = \frac{1}{\sqrt{2}}\left[a_1^\dagger(\boldsymbol{p},s) + \mathrm{i}a_2^\dagger(\boldsymbol{p},s)\right]. \end{aligned} \tag{C2}$$

The action of a Dirac field is the sum of the actions of its Majorana fields

$$-\overline{\psi}(\gamma^a\partial_a + m)\psi = -\frac{1}{2}\overline{\psi}_{M1}(\gamma^a\partial_a + m)\psi_{M1} - \frac{1}{2}\overline{\psi}_{M2}(\gamma^a\partial_a + m)\psi_{M2} \tag{C3}$$

in which $\overline{\psi} = \mathrm{i}\psi^\dagger\gamma^0 = \psi^\dagger\beta$ and $a = 0, 1, 2, 3$. The cross-terms

$$-\mathrm{i}\frac{1}{2}\overline{\psi}_{M1}(\gamma^a\partial_a + m)\psi_{M2} + \mathrm{i}\frac{1}{2}\overline{\psi}_{M2}(\gamma^a\partial_a + m)\psi_{M1} \tag{C4}$$

vanish if we integrate by parts and drop surface terms because the fields anticommute

$$\{\psi_{M1}^\dagger(x), \psi_{M2}(y)\} = 0 \quad \text{and} \quad \{\psi_{M2}^\dagger(x), \psi_{M1}(y)\} = 0, \tag{C5}$$

because they obey the Majorana condition (A7), and because the matrix $\gamma^2\gamma^0$ is antisymmetric, while the matrices $\gamma^2\gamma^0\gamma^a$ for $a = 0, 1, 2, 3$ are symmetric

$$(\gamma^2\gamma^0)_{\alpha\beta} = -(\gamma^2\gamma^0)_{\beta\alpha} \quad \text{and} \quad (\gamma^2\gamma^0\gamma^a)_{\alpha\beta} = (\gamma^2\gamma^0\gamma^a)_{\beta\alpha}. \tag{C6}$$

By using the formulas (7) for the gamma matrices, we may write the action for a single Majorana field as

$$\begin{aligned} -\frac{1}{2}\overline{\psi}_M(\gamma^a\partial_a + m)\psi_M &= -\frac{1}{2}\mathrm{i}\psi_M^\dagger\gamma^0(\gamma^a\partial_a + m)\psi_M \\ &= -\frac{1}{2}\mathrm{i}\psi_M^\dagger(-\partial_0 + \gamma^0\boldsymbol{\gamma}\cdot\,\nabla + m\gamma^0)\psi_M. \end{aligned} \tag{C7}$$

Like a Dirac field, a 4-component Majorana field is composed of two 2-component fields [14]

$$\psi_M = \begin{pmatrix} \ell \\ r \end{pmatrix}, \tag{C8}$$

in which ℓ is left handed and r is right handed. In terms of ℓ and r, the action (C7) of the Majorana field may be written as [13, 14]

$$\begin{aligned} -\frac{1}{2}\overline{\psi}_M(\gamma^a\partial_a + m)\psi_M &= \frac{1}{2}\mathrm{i}\begin{pmatrix} \ell^\dagger & r^\dagger \end{pmatrix}\begin{pmatrix} \partial_0 - \boldsymbol{\sigma}\cdot\nabla & \mathrm{i}m \\ \mathrm{i}m & \partial_0 + \boldsymbol{\sigma}\cdot\nabla \end{pmatrix}\begin{pmatrix} \ell \\ r \end{pmatrix} \\ &= \frac{1}{2}\mathrm{i}\ell^\dagger(\partial_0 - \boldsymbol{\sigma}\cdot\nabla)\ell + \frac{1}{2}\mathrm{i}r^\dagger(\partial_0 + \boldsymbol{\sigma}\cdot\nabla)r \\ &\quad - \frac{1}{2}m(\ell^\dagger r + r^\dagger \ell). \end{aligned} \tag{C9}$$

In this notation, the Majorana condition (A7)

$$\begin{pmatrix} \ell^* \\ r^* \end{pmatrix} = -\mathrm{i}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \ell \\ r \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sigma^2 r \\ \mathrm{i}\sigma^2 \ell \end{pmatrix} \tag{C10}$$

tells us that $\ell = -\mathrm{i}\sigma^2 r^*$ and $r = \mathrm{i}\sigma^2\ell^*$, or more simply that $\ell_1 = -r_2^*$ and $\ell_2 = r_1^*$. So one can write the action (C9) of a Majorana field entirely in terms of ℓ

$$\begin{aligned} -\frac{1}{2}\overline{\psi}_M(\gamma^a\partial_a + m)\psi_M &= \frac{1}{2}\mathrm{i}\ell^\dagger(\partial_0 - \boldsymbol{\sigma}\cdot\nabla)\ell + \frac{1}{2}\mathrm{i}\ell^\mathsf{T}(\partial_0 - \boldsymbol{\sigma}^*\cdot\nabla)\ell^* \\ &\quad - \frac{1}{2}\mathrm{i}m(\ell^\dagger\sigma^2\ell^* - \ell^\mathsf{T}\sigma^2\ell) \end{aligned} \tag{C11}$$

or entirely in terms of r

$$\begin{aligned} -\frac{1}{2}\overline{\psi}_M(\gamma^a\partial_a + m)\psi_M &= \frac{1}{2}\mathrm{i}r^\mathsf{T}(\partial_0 + \boldsymbol{\sigma}^*\cdot\nabla)r^* + \frac{1}{2}\mathrm{i}r^\dagger(\partial_0 + \boldsymbol{\sigma}\cdot\nabla)r \\ &\quad - \frac{1}{2}\mathrm{i}m(r^\mathsf{T}\sigma^2 r - r^\dagger\sigma^2 r^*). \end{aligned} \tag{C12}$$

The action (C9) takes simpler forms when we integrate by parts, anticommute the fields, and drop both the surface terms and an infinite constant

$$\begin{aligned} -\frac{1}{2}\overline{\psi}_M(\gamma^a\partial_a + m)\psi_M &= \mathrm{i}\ell^\dagger(\partial_0 - \boldsymbol{\sigma}\cdot\nabla)\ell - \frac{1}{2}\mathrm{i}m(\ell^\dagger\sigma^2\ell^* - \ell^\mathsf{T}\sigma^2\ell) \\ &= \mathrm{i}r^\dagger(\partial_0 + \boldsymbol{\sigma}\cdot\nabla)r - \frac{1}{2}\mathrm{i}m(r^\mathsf{T}\sigma^2 r - r^\dagger\sigma^2 r^*). \end{aligned} \tag{C13}$$

Under a Lorentz transformation L, the fields $\ell(x)$ and $r(x)$ transform as

$$\begin{aligned} U(L)\,\ell(x)\,U^{-1}(L) &= D^{(1/2,0)}(L^{-1})\,\ell(Lx) \\ U(L)\,r(x)\,U^{-1}(L) &= D^{(0,1/2)}(L^{-1})\,r(Lx) \end{aligned} \tag{C14}$$

in which the unitary operator $U(L) \equiv U(L(\boldsymbol{\theta}, \boldsymbol{\lambda}))$ and the complex, unimodular 2×2 matrices of unit determinant

$$D^{(1/2,0)}(L(\boldsymbol{\theta}, \boldsymbol{\lambda})) = \mathrm{e}^{-(\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{\theta})\cdot\boldsymbol{\sigma}/2} \quad \text{and} \quad D^{(0,1/2)}(L(\boldsymbol{\theta}, \boldsymbol{\lambda})) = \mathrm{e}^{(\boldsymbol{\lambda}-\mathrm{i}\boldsymbol{\theta})\cdot\boldsymbol{\sigma}/2} \tag{C15}$$

both represent the Lorentz transformation

$$L(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \mathrm{e}^{\boldsymbol{\theta}\cdot\boldsymbol{R}+\boldsymbol{\lambda}\cdot\boldsymbol{B}} \tag{C16}$$

where

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C17}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{C18}$$

For small $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, the Lorentz transformation $L(\boldsymbol{\theta}, \boldsymbol{\lambda})$ changes $t, \boldsymbol{x}$ to

$$\begin{aligned} t' &\simeq t + \boldsymbol{\lambda} \cdot \boldsymbol{x} \\ \boldsymbol{x}' &\simeq \boldsymbol{x} + t\boldsymbol{\lambda} + \boldsymbol{\theta} \wedge \boldsymbol{x} \end{aligned} \tag{C19}$$

in which $\wedge \equiv \times$ means cross-product.

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References

[1] Ryder L 1996 *Quantum Field Theory* 2nd edn (Cambridge: Cambridge University Press)
[2] Kaku M 1993 *Quantum Field Theory: A Modern Introduction* (Oxford: Oxford University Press)
[3] Schwartz M D 2014 *Quantum Field Theory and the Standard Model* (Cambridge: Cambridge University Press) pp 189–94
[4] Srednicki M 2007 *Quantum Field Theory* (Cambridge: Cambridge University Press) pp 237–43
[5] Peskin M E and Schroeder D V 1995 *An Introduction to Quantum Field Theory* (Boulder: Westview Press) pp 803–4
[6] Coleman S 2018 *Lectures of Sidney Coleman on Quantum Field Theory* ed B G-g Chen, D Derbes, D Griffiths, B Hill, R Sohn and Y-S Ting (Hackensack: WSP)
[7] Mandl F and Shaw G 1985 *Quantum Field Theory* 2nd edn (New York: Wiley)
[8] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) p 224
[9] Cahill P and Cahill K 2006 Learning about spin-one-half fields *Eur. J. Phys.* **27** 29
Cahill P and Cahill K 2007 *Eur. J. Phys.* **28** 145 Erratum
[10] Sakurai J J 1994 *Advanced Quantum Mechanics* 1st edn (Reading, MA: Addison Wesley Longman) ch 3
[11] Weinberg S 1995 *The Quantum Theory of Fields, Vol I Foundations* (Cambridge: Cambridge University Press) p 216

[12] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) pp 213–29
[13] Cahill K 2019 *Physical Mathematics* 2nd edn (Cambridge: Cambridge University Press) pp 435–47
[14] Dreiner H K, Haber H E and Martin S P 2010 Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry *Phys. Rep.* **494** 1
[15] Cahill P and Cahill K 2006 Learning about spin-one-half fields *Eur. J. Phys.* **27** 29
[16] Cahill K 2019 *Physical Mathematics* 2nd edn (Cambridge: Cambridge University Press) pp 256–7
[17] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) pp 191–229
[18] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) pp 177–225
[19] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) pp 177–226
[20] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) p 177
[21] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) pp 191–7
[22] Wigner E 1939 On unitary representations of the inhomogeneous Lorentz group *Ann. Math.* **40** 149
[23] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge: Cambridge University Press) pp 62–9