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Neutrinos Are Nearly Dirac Fermions

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Abstract

Neutrino masses and mixings are analyzed in terms of left-handed fields and a 6×6 complex symmetric mass matrix $\mathcal{M}$ whose singular values are the neutrino masses. An angle x_ν characterizes the kind of the neutrinos, with $x_\nu = 0$ for Dirac neutrinos and $x_\nu = \pi/2$ for Majorana neutrinos. If $x_\nu = 0$, then baryon-minus-lepton number is conserved. When $x_\nu \approx 0$, the six neutrino masses coalesce into three nearly degenerate pairs. Thus the smallness of the differences in neutrino masses exhibited in the solar and atmospheric neutrino experiments and the stringent limits on neutrinoless double beta decay are naturally explained if $B - L$ is approximately conserved and neutrinos are nearly Dirac fermions. If one sets $\sin x_\nu = 0.003$, suppresses inter-generational mixing, and imposes a quark-like mass hierarchy, then one may fit the essential features of the solar, reactor, and atmospheric neutrino data with otherwise random mass matrices $\mathcal{M}$ in the eV range.

This $B - L$ model leads to these predictions: neutrinos oscillate mainly between flavor eigenfields and sterile eigenfields, and so the probabilities of the appearance of neutrinos or antineutrinos are very small; neutrinos may well be of cosmological importance; in principle the disappearance of ν_τ should be observable; and $0\nu\beta\beta$ decay is suppressed by an extra factor of 10^{-5} and hence will not be seen in the Heidelberg/Moscow, IGEX, GENIUS, or CUORE experiments.

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1 Introduction

There are three major sets of experimental results that shed light on neutrino masses and mixings. Solar-neutrino and reactor experiments have shown that the electron neutrino ν_e couples to two nearly degenerate mass eigenfields with $10^{-10} \lesssim m_1^2 - m_2^2 \lesssim 10^{-3}\,\mathrm{eV}^2$. The atmospheric-neutrino experiments have shown that the muon neutrino ν_μ and antineutrino $\bar{\nu}_\mu$ couple to two nearly degenerate mass eigenfields with $10^{-3} \lesssim m_3^2 - m_4^2 \lesssim 10^{-2}\,\mathrm{eV}^2$. Double-beta-decay experiments have placed very stringent limits on the rates of neutrinoless double beta decay.

The current wisdom on neutrinos is that they have very small masses and that the tiny mass differences experimentally observed are due to their small masses. It is also generally believed that the smallness of the masses of the neutrinos arises from the seesaw mechanism [1], which involves huge Majorana masses. It is also thought that CKM-like mixings explain the solar and atmospheric data.

The purpose of this paper is to present a general discussion of neutrino masses and mixings and a rather different explanation of the experimental facts. It is argued that the neutrino mass matrix is a 6×6 complex symmetric matrix, that the moduli of the Majorana mass terms are small compared to those of the Dirac mass terms, that neutrinos propagate as six mass eigenfields with masses that form three nearly degenerate pairs, and that the oscillations observed in the solar and atmospheric experiments are mainly between members of these pairs. Neutrino masses and mixings have been discussed by several authors [2, 3], and the model introduced by Geiser [4] is an important example of the class of models to be developed in what follows.

Because left-handed fields can mix only with other left-handed fields, neutrino masses are best analyzed in terms of left-handed fields. There are six such fields, the three left-handed flavor eigenfields ν_e, ν_μ and ν_τ, and the charge conjugates of the three right-handed neutrinos, which are sterile. In section 2 the neutrino mass matrix is exhibited as a 6×6 complex symmetric matrix $\mathcal{M}$. In general this matrix is not normal, but like every matrix it admits a singular-value decomposition $\mathcal{M} = UMV^\dagger$ in which the matrices U and V are unitary and the matrix M is real, nonnegative, and diagonal [5]. Moreover because the matrix $\mathcal{M}$ is symmetric it follows from a theorem of

Takagi's [6, 7] that one may always choose $V = U^*$. In section 3, Takagi's factorization $\mathcal{M} = UMU^\top$ is used to diagonalize the action and to show that the singular values of the neutrino mass matrix $\mathcal{M}$ or equivalently the diagonal elements of the matrix M are the six neutrino masses m_j and that the unitary matrix U describes the neutrino mixings.

In section 4 the LEP measurement of the number of light neutrino flavors is related to the mixing matrix U. Section 5 describes a cosmological constraint on the masses of light, stable neutrinos. Section 6 expresses the neutrino oscillation probabilities in terms of the mixing matrix U and the masses m_j and briefly discusses the results of the solar, reactor, and atmospheric neutrino experiments.

In section 7 an angle x_ν is introduced that quantifies the extent to which neutrinos are Dirac fermions or Majorana fermions. Dirac neutrinos have $x_\nu = 0$, and Majorana neutrinos have $x_\nu = \pi/2$. If all Majorana mass terms vanish, that is if $x_\nu = 0$, then the standard model conserves baryon-minus-lepton number, $B-L$, which is a global $U(1)$ symmetry. It is therefore natural in the sense of 't Hooft [8] to assume that $x_\nu \approx 0$ so that this symmetry is only slightly broken. The neutrinos then are nearly Dirac fermions and their masses coalesce into three pairs of almost degenerate masses. Thus the approximate conservation of $B - L$ explains the tiny mass differences seen in the solar and atmospheric neutrino experiments without requiring the neutrino masses to be absurdly small. There are three tiny mass differences, of which one explains the solar experiments and another the atmospheric ones, and three unconstrained mass differences, which can lie in the eV range. If one sets $\sin x_\nu = 0.003$, suppresses inter-generational mixing, and imposes a quark-like mass hierarchy, then one may fit the essential features of the solar, reactor, and atmospheric neutrino data with otherwise random mass matrices $\mathcal{M}$ in the eV range. Thus neutrinos easily can have masses that saturate the cosmological bound of about 8 eV. Moreover because neutrinos are almost Dirac fermions, neutrinoless double beta decay is suppressed by an extra factor $\sim \sin^2 x_\nu \, \sin^2 y_\nu \lesssim 10^{-5}$, where y_ν is a second neutrino angle, and is very slow, with lifetimes in excess of 2×10^{27} years.

An appendix outlines an efficient way of performing the singular-value decomposition of the mass matrix $\mathcal{M}$ by means of a call to the LAPACK [9] subroutine ZGESVD [10].

This $B-L$ model of neutrino masses and mixings leads to these predictions about future experiments:

- The three flavor neutrinos oscillate mainly into the conjugates of the right-handed fields, which are sterile, but the mass differences among the three pairs of nearly degenerate masses can lie in the eV range. The probabilities of the appearance of neutrinos or antineutrinos are small, as shown by LSND and KARMEN2, but MiniBooNE has a reasonable chance of seeing the appearance of $\bar{\nu}_\mu \to \bar{\nu}_e$.
- Because neutrino masses are not required to be nearly as small as the solar and atmospheric mass differences might suggest, they may well be of cosmological significance.
- If a suitable experiment can be designed, it should be possible to see the tau neutrino disappear.
- The rate of neutrinoless double beta decay is suppressed by an extra factor of $\sim \sin^2 x_\nu \sin^2 y_\nu \lesssim 10^{-5}$ and hence will not be seen in the Heidelberg/Moscow, IGEX, GENIUS, or CUORE experiments.

2 Mass Terms and Mass Matrices

There are three potential sources of neutrino masses. One source is Dirac mass terms, which require the existence of right-handed neutrino fields. The second source of neutrino masses is Majorana mass terms composed of left-handed neutrino fields; these mass terms require a triplet of Higgs bosons, break $B-L$, and drive neutrinoless double beta decay. The third source is Majorana mass terms composed of right-handed neutrino fields; these terms also break $B-L$ but do not affect neutrinoless double beta decay, at least in leading order. Inasmuch as quarks and charged leptons come in both left-handed and right-handed fields, the existence of right-handed neutrino fields is reasonable. And because the Higgs sector is still unknown, the existence of Higgs bosons that transform as a triplet under $SU(2)_L$ remains possible. So in this paper we shall consider all three kinds of mass terms.

Because left- and right-handed fields transform differently under Lorentz boosts, they cannot mix. It is therefore convenient to write the action density exclusively in terms of two-component, left-handed fields. The two-component, left-handed neutrino flavor eigenfields ν_e, ν_μ, ν_τ will be denoted ν_i, for $i = e, \mu, \tau$. The two-component, left-handed fields that are the charge conjugates of the putative right-handed neutrino fields $n_{re}, n_{r\mu}, n_{r\tau}$ will be denoted $n_i = -i\sigma_2\, n_{ri}^*$ for $i = e, \mu, \tau$ in which hermitian conjugation without transposition is denoted by an asterisk, and σ_2 is the second Pauli spin matrix.

For two left-handed fields ψ and χ, there is only one mass term, and its hermitian form is

$$i\, m\chi^\top \sigma_2 \psi - i\, m^* \psi^\dagger \sigma_2 \chi^* = i \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \left(m\, \chi_\alpha \sigma_{2\alpha\beta} \psi_\beta - m^*\, \psi_\alpha^* \sigma_{2\alpha\beta} \chi_\beta^* \right), \qquad (1)$$

which incidentally is invariant under a Lorentz transformation $g \in SL(2, C)$ because the antisymmetric matrix σ_2 converts two factors of g into $\det(g) = 1$. If the field χ is the charge conjugate of a right-handed field that carries the same conserved quantum numbers as the field ψ, then this mass term (1) is called a Dirac mass term. On the other hand, if the fields ψ and χ carry the same quantum numbers or no conserved quantum numbers at all, then the mass term (1) is called a Majorana mass term. Since electric charge is conserved, only neutral fields can have Majorana mass terms.

The six left-handed neutrino fields ν_i, n_i for $i = 1, 2, 3$ can have three kinds of mass terms. The fields ν_i and n_j can form Dirac mass terms

$$iD_{ij} \nu_i^\top \sigma_2 n_j - iD_{ij}^* n_j^\dagger \sigma_2 \nu_i^* \qquad (2)$$

in which the D_{ij} are complex numbers. In a minimal extension of the standard model, the D_{ij} are proportional to the mean value in the vacuum of the neutral component of the Higgs field. The fields n_i and n_j can form Majorana mass terms

$$iE_{ij} n_i^\top \sigma_2 n_j - iE_{ij}^* n_j^\dagger \sigma_2 n_i^*. \qquad (3)$$

Within the standard model, the complex numbers E_{ij} are simply numbers; in a more unified theory, they might be the mean values in the vacuum of neutral components of Higgs bosons. The fields ν_i and ν_j can also form Majorana mass terms

$$i F_{ij} \nu_i^{\top} \sigma_2 \nu_j - i F_{ij}^{*} \nu_j^{\dagger} \sigma_2 \nu_i^{*}. \tag{4}$$

In a minimal extension of the standard model, the complex numbers F_{ij} might be proportional to the mean values in the vacuum of the neutral component of a new Higgs triplet $h_{ab} = h_{ba}$ that transforms as

$$h'_{ab}(x) = h_{a'b'} g^{-1}_{a'a}(x) g^{-1}_{b'b}(x) \tag{5}$$

in which the 2×2 matrix $g_{aa'}(x)$ is the one that transforms the three doublets

$$L_i(x) = \begin{pmatrix} \nu_i(x) \\ e_i(x) \end{pmatrix} \qquad \text{as} \qquad L_i(x)' = g(x) L_i(x) \tag{6}$$

where $e_i = e, \mu, \tau$ for $i = 1, 2, 3$. Such a triplet Higgs multiplet would allow a term like

$$\sum_{a,b=1}^{2} h_{ab}(x) L_{ia}(x)^{\top} \sigma_2 \, L_{jb}(x) \tag{7}$$

to remain invariant under $SU_L(2) \otimes U(1)_Y$ gauge transformations and so to contribute a mass term to the action density.

Since σ_2 is antisymmetric and since any two fermion fields χ and ψ anticommute, it follows that

$$\chi^{\top} \sigma_2 \psi = \psi^{\top} \sigma_2 \chi \quad \text{and} \quad \chi^{\dagger} \sigma_2 \psi^{*} = \psi^{\dagger} \sigma_2 \chi^{*}, \tag{8}$$

which incidentally shows that a singlet Higgs field, $h_{ab} = -h_{ba}$, would lead to $F_{ij} = 0$. This symmetry relation (8) implies that the 3×3 complex matrices E and F are symmetric

$$E^{\top} = E \quad \text{and} \quad F^{\top} = F \tag{9}$$

and that

$$i D_{ij} n_j^{\top} \sigma_2 \nu_i = i D_{ij} \nu_i^{\top} \sigma_2 n_j. \tag{10}$$

Let us introduce the 6×6 complex matrix $\mathcal{M}$

$$\mathcal{M} = \begin{pmatrix} F & D \\ D^{\top} & E \end{pmatrix} \tag{11}$$

which by (9) is symmetric. Then with the six-vector N of left-handed neutrino fields

$$N = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ n_e \\ n_\mu \\ n_\tau \end{pmatrix}, \tag{12}$$

we may gather the mass terms into the matrix expression

$$\frac{i}{2} N^\top \mathcal{M} \sigma_2 N - \frac{i}{2} N^\dagger \mathcal{M}^* \sigma_2 N^*. \tag{13}$$

Actually since each of the six fields in N has two components, the six-vector N has twelve components. The use of twelve-component fields has been advocated by Rosen [11] and more recently by Starkman and Stojkovic [12].

The complex symmetric mass matrix $\mathcal{M}$ is not normal unless the positive hermitian matrix $\mathcal{M}\mathcal{M}^\dagger$ is real because the commutator $[\mathcal{M}, \mathcal{M}^\dagger]$ is twice its imaginary part:

$$[\mathcal{M}, \mathcal{M}^\dagger] = 2i \, \Im m \left(\mathcal{M}\mathcal{M}^\dagger \right). \tag{14}$$

When the mass matrix is itself real, then it is also normal and may be diagonalized by an orthogonal transformation. But in general $\mathcal{M}$ is neither real nor normal.

Yet like every matrix, the mass matrix $\mathcal{M}$ possesses a singular-value decomposition [5, 9]

$$\mathcal{M} = U M V^\dagger \tag{15}$$

in which the 6×6 matrices U and V are unitary and the 6×6 matrix M is real and diagonal

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6 \end{pmatrix} \tag{16}$$

with singular values $m_j \geq 0$. These singular values will turn out to be the masses of the six neutrinos. The columns of the unitary matrix U are the left singular vectors of $\mathcal{M}$; they are the eigenvectors of the matrix $\mathcal{M}\,\mathcal{M}^\dagger$ with eigenvalues m_j^2

$$\mathcal{M}\,\mathcal{M}^\dagger\,U = U\,M^2. \tag{17}$$

The columns of the unitary matrix V are the right singular vectors of $\mathcal{M}$; they are the eigenvectors of the matrix $\mathcal{M}^\dagger\,\mathcal{M}$ with eigenvalues m_j^2

$$\mathcal{M}^\dagger\,\mathcal{M}\,V = V\,M^2. \tag{18}$$

We have not yet made use of the fact that the complex matrix $\mathcal{M}$ is symmetric. It is a general mathematical theorem [6] due to Takagi [7] that for every symmetric complex matrix Z there is a unitary matrix X and a real diagonal matrix D with nonnegative elements such that $Z = XDX^\top$. The diagonal matrix elements D_{jj} are the singular values of the matrix Z. Thus the symmetric complex matrix $\mathcal{M}$ may be factored by a unitary matrix W

$$\mathcal{M} = WMW^\top \tag{19}$$

where M is the matrix of masses (16). The columns of W are the eigenvectors of the matrix $\mathcal{M}\,\mathcal{M}^\dagger$ with eigenvalues m_j^2

$$\mathcal{M}\,\mathcal{M}^\dagger\,W = W\,M^2. \tag{20}$$

Thus by (17) the columns of W are the same as the columns of the unitary matrix U of the singular-value decomposition (15) apart from over-all phase factors. Similarly the columns of W^* are the eigenvectors of the matrix $\mathcal{M}^\dagger\,\mathcal{M}$ with eigenvalues m_j^2

$$\mathcal{M}^\dagger\,\mathcal{M}\,W^* = W^*\,M^2, \tag{21}$$

and so by (18) the columns of W^* are the same as the columns of the unitary matrix V of the singular-value decomposition (15) apart from over-all phases. For our purposes the import of Takagi's theorem is that the singular-value decomposition of the symmetric complex matrix $\mathcal{M} = UMV^\dagger$ may be achieved with $U = W$ and $V = W^*$ so that

$$\mathcal{M} = U\,M\,U^\top, \tag{22}$$

which we may call Takagi's factorization.

3 Field Equations, Masses, and Mixings

The free, kinetic action density of a two-component left-handed spinor ψ is $i\psi^\dagger(\partial_0 - \vec{\sigma}\cdot\nabla)\psi$. Thus by including the mass terms (13), one may write the free action density of the six left-handed neutrino fields N as

$$\mathcal{L}_0 = iN^\dagger(\partial_0 - \vec{\sigma}\cdot\nabla)N + \frac{i}{2}N^\top \mathcal{M}\sigma_2 N - \frac{i}{2}N^\dagger \mathcal{M}^*\sigma_2 N^* \tag{23}$$

or with Takagi's factorization (22) of the mass matrix as

$$\mathcal{L}_0 = iN^\dagger(\partial_0 - \vec{\sigma}\cdot\nabla)N + \frac{i}{2}N^\top U\, M\sigma_2\, U^\top N - \frac{i}{2}N^\dagger U^*\, M\sigma_2 U^\dagger N^* \tag{24}$$

in which M is the real 6×6 diagonal matrix (16) with nonnegative entries m_j. Let us define the six-vector of fields

$$N_m = U^\top N \quad \text{or} \quad \nu_{m_i} = \sum_{j=1}^{6} U_{ji} N_j. \tag{25}$$

Since the 6×6 matrix U is unitary, we have

$$N = U^*\, N_m \qquad \text{and} \qquad N^\top = N_m^\top\, U^\dagger \tag{26}$$

as well as

$$N^* = U\, N_m^* \qquad \text{and} \qquad N^\dagger = N_m^\dagger\, U^\top. \tag{27}$$

Thus we may write $\mathcal{L}_0$ in the form

$$\begin{aligned} \mathcal{L}_0 \;=\;& iN_m^\dagger\, U^\top\,(\partial_0 - \vec{\sigma}\cdot\nabla)\; U^*\, N_m \\ & + \frac{i}{2}N_m^\top\, U^\dagger\, U M\sigma_2 U^\top\, U^*\, N_m - \frac{i}{2}N_m^\dagger\, U^\top\, U^* M\sigma_2 U^\dagger\, U\, N_m^* \end{aligned} \tag{28}$$

or since $U^\top\, U^* = I = U^\dagger\, U$

$$\mathcal{L}_0 = iN_m^\dagger\;(\partial_0 - \vec{\sigma}\cdot\nabla)\; N_m + \frac{i}{2}N_m^\top\, M\sigma_2\, N_m - \frac{i}{2}N_m^\dagger\, M\sigma_2\, N_m^*. \tag{29}$$

The action density therefore has the diagonal form

$$\mathcal{L}_0 = \sum_{i=1}^{6}\left[i\,\nu_{m_i}^\dagger\;(\partial_0 - \vec{\sigma}\cdot\nabla)\;\nu_{m_i} + \frac{i}{2}m_i\left(\nu_{m_i}^\top\,\sigma_2\,\nu_{m_i} - \nu_{m_i}^\dagger\,\sigma_2\,\nu_{m_i}^*\right)\right]. \tag{30}$$

The fields ν_{m_i} are the normal modes of the theory. They propagate according to the equation of motion

$$(\partial_0 - \vec{\sigma} \cdot \nabla)\ \nu_{m_i} = m_i\, \sigma_2\, \nu^*_{m_i} \tag{31}$$

in which the asterisk represents complex or hermitian conjugation without transposition. By taking the hermitian adjoint of this equation without transposing the matrices and vectors (or equivalently by taking the hermitian adjoint of the equation and then re-transposing the matrices and vectors), one has

$$(\partial_0 - \vec{\sigma}^* \cdot \nabla)\ \nu^*_{m_i} = m_i\, \sigma^*_2\, \nu_{m_i}. \tag{32}$$

The rules $\sigma_2 \vec{\sigma}^* \sigma_2 = -\vec{\sigma}$, $(\sigma_2)^2 = 1$, and $\sigma^{2*} = -\sigma_2$ now imply that the equation of motion for $\nu^*_{m_i}$ is

$$(\partial_0 + \vec{\sigma} \cdot \nabla)\, \sigma_2 \nu^*_{m_i} = -m_i\, \nu_{m_i}. \tag{33}$$

Applying $(\partial_0 + \vec{\sigma} \cdot \nabla)$ to the field equation (31) for ν_{m_i} and then using the field equation (33) for $\nu^*_{m_i}$, we find that

$$\left(\Box - m_i^2\right) \nu_{m_i} = 0. \tag{34}$$

Thus the diagonal elements m_i of the real mass matrix M, which are the singular values of the mass matrix $\mathcal{M}$, are the neutrino masses, and the eigenfield of mass m_i is ν_{m_i} as given by (25). A neutrino of mass m_i described by the two-component field ν_{m_i} must have two helicities or spin states and so must be its own antiparticle. The two-component field ν_{m_i} is not hermitian, however; we'd need to use four components to make this field hermitian.

Since the mass eigenfields ν_{m_i} are the normal modes of the theory, it is worth recoding their expansions

$$\nu_{m_i}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \left[u(\vec{p}, s, m_i) e^{ipx} a(\vec{p}, s, m_i) + v(\vec{p}, s, m_i) e^{-ipx} a^\dagger(\vec{p}, s, m_i) \right] \tag{35}$$

in which $px = \vec{p} \cdot \vec{x} - et$. In terms of the normalization factor

$$n_i = \frac{1}{2\sqrt{e(e + m_i)}}, \tag{36}$$

the two-component spinors u and v are

$$u(\vec{p}, \tfrac{1}{2}, m_i) = n_i \begin{pmatrix} m_i + e - p_3 \\ -p_1 - ip_2 \end{pmatrix} \quad u(\vec{p}, -\tfrac{1}{2}, m_i) = n_i \begin{pmatrix} -p_1 + ip_2 \\ m_i + e + p_3 \end{pmatrix} \tag{37}$$

and

$$v(\vec{p}, \tfrac{1}{2}, m_i) = n_i \begin{pmatrix} -p_1 + ip_2 \\ m_i + e + p_3 \end{pmatrix} \quad v(\vec{p}, -\tfrac{1}{2}, m_i) = n_i \begin{pmatrix} p_3 - m_i - e \\ p_1 + ip_2 \end{pmatrix}. \tag{38}$$

They satisfy the spin sums

$$\sum_s u(\vec{p}, s, m_i)\, u^\dagger(\vec{p}, s, m_i) = \sum_s v(\vec{p}, s, m_i)\, v^\dagger(\vec{p}, s, m_i) = \frac{1}{2e}(e - \vec{p} \cdot \vec{\sigma}) \tag{39}$$

and

$$\sum_s u(\vec{p}, s, m_i) v^\top(\vec{p}, s, m_i) = \frac{im_i}{2e}\, \sigma_2. \tag{40}$$

For momentum $\vec{p} = p\hat{z}$ in the z-direction with $p \gg m_i$, these spinors reduce to

$$u(\vec{p}, \tfrac{1}{2}, m_i) \approx \frac{m_i}{2p} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u(\vec{p}, -\tfrac{1}{2}, m_i) \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{41}$$

and

$$v(\vec{p}, \tfrac{1}{2}, m_i) \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v(\vec{p}, -\tfrac{1}{2}, m_i) \approx -\frac{m_i}{2p} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{42}$$

which shows that the field $\nu_{m_i}(x)$ primarily deletes neutrinos of negative helicity and adds neutrinos of positive helicity.

In terms of the flavor eigenfields ν_i and n_i, the eigenfield of mass m_i is

$$\nu_{m_j} = \sum_{i=1}^{6} U_{ij} N_i = \sum_{i=1}^{3} U_{ij} \nu_i + \sum_{i=1}^{3} U_{(i+3)j} n_i. \tag{43}$$

The flavor eigenfields N are given by

$$N = U^* N_m. \tag{44}$$

In particular, the three left-handed fields ν_i for $i = e, \mu, \tau$ are linear combinations of the six mass eigenfields ν_{m_j}

$$\nu_i = \sum_{j=1}^{6} U_{ij}^* \nu_{m_j} \tag{45}$$

and not simply linear combinations of three mass eigenfields. The three active (right-handed) antineutrino fields ν_i^* for $i = e, \mu, \tau$ are similarly

$$i\sigma_2 \nu_i^* = \sum_{j=1}^{6} U_{ij} i\sigma_2 \nu_{m_j}^* . \tag{46}$$

The matrix that expresses the flavor eigenfields in terms of the mass eigenfields is a 3×6 matrix that is half of a 6×6 unitary matrix, not a 3×3 unitary matrix. The three sterile left-handed fields n_i for $i = e, \mu, \tau$ are also linear combinations of the six mass eigenfields ν_{m_j}

$$n_i = \sum_{j=1}^{6} U_{(i+3)j}^* \nu_{m_j} . \tag{47}$$

In view of the relation (45) between the flavor eigenfields and the mass eigenfields, the action density for neutrino interactions in four-component form [13]

$$\mathcal{L}' = \frac{-ie}{4\sin\theta_W} \sum_{i=1}^{3} \left\{ \sqrt{2}\left[\bar{e}_i \not{W}(1+\gamma_5)\nu_i + \bar{\nu}_i \not{W}^*(1+\gamma_5)e_i\right] - \frac{\bar{\nu}_i \not{Z}(1+\gamma_5)\nu_i}{\cos\theta_W} \right\} \tag{48}$$

may be written as

$$\begin{aligned} \mathcal{L}' \;=\; & \frac{-ie}{4\sin\theta_W} \sum_{i=1}^{3} \left\{ \sqrt{2} \left[\bar{e}_i \not{W}(1+\gamma_5) \sum_{j=1}^{6} U_{ij}^* \nu_{m_j} + \sum_{k=1}^{6} U_{ik} \bar{\nu}_{m_k} \not{W}^*(1+\gamma_5) e_i \right] \right. \\ & \left. - \sum_{j=1}^{6} \sum_{k=1}^{6} U_{ik} U_{ij}^* \frac{\bar{\nu}_{m_k} \not{Z}(1+\gamma_5)\nu_{m_j}}{\cos\theta_W} \right\} . \end{aligned} \tag{49}$$

In two-component form, this is

$$\begin{aligned} \mathcal{L}' \;=\; & \frac{e}{2\sin\theta_W} \sum_{i=1}^{3} \left\{ \sqrt{2} \left[e_i^\dagger (W^0 + \vec{W}\cdot\vec{\sigma}) \sum_{j=1}^{6} U_{ij}^* \nu_{m_j} \right.\right. \\ & \left. + \sum_{k=1}^{6} U_{ik} \nu_{m_k}^\dagger (W^{0*} + \vec{W}^* \cdot \vec{\sigma}) e_i \right] \\ & \left. - \sum_{j=1}^{6} \sum_{k=1}^{6} U_{ik} U_{ij}^* \frac{\nu_{m_k}^\dagger (Z^0 + \vec{Z}\cdot\vec{\sigma}) \nu_{m_j}}{\cos\theta_W} \right\} \end{aligned} \tag{50}$$

in which all fields are left-handed two-component spinors.

4 LEP

The four LEP measurements of the invisible partial width of the Z impose upon the number of light neutrino types the constraint [14]

$$N_\nu = 2.984 \pm 0.008. \tag{51}$$

It follows from the action density (49) that the amplitude for the Z production of two neutrinos ν_{m_j} and ν_{m_k} to lowest order is

$$A(\nu_{m_j}, \nu_{m_k}) \propto \sum_{i=1}^{3} U_{ik} U_{ij}^*, \tag{52}$$

and therefore that the cross-section for that process is

$$\sigma(\nu_{m_j}, \nu_{m_k}) \propto |\sum_{i=1}^{3} U_{ik} U_{ij}^*|^2. \tag{53}$$

The measurement (51) of the number N_ν of light neutrino species thus implies that the sum over the light-mass eigenfields is

$$\sum_{j,k \text{ light}} |\sum_{i=1}^{3} U_{ik} U_{ij}^*|^2 = 2.984 \pm 0.008. \tag{54}$$

This constraint on the 6×6 unitary matrix V is quite well satisfied if all six neutrino masses are light. For in this all-light scenario, the sum is

$$\sum_{j,k=1}^{6} \sum_{i=1}^{3} \sum_{i'=1}^{3} U_{ik} U_{ij}^* U_{i'k}^* U_{i'j} = \sum_{i=1}^{3} \sum_{i'=1}^{3} \delta_{ii'} \delta_{ii'} = \sum_{i=1}^{3} 1 = 3 \approx 2.984 \pm 0.008. \tag{55}$$

But the constraint (51) will also be satisfied if the three flavor eigenfields ν_i couple only to light-mass eigenfields ν_{m_j}, for in this case the matrix elements U_{ij}^* between flavor i and heavy mass m_j vanish, and one may extend the sum over the light-mass eigenfields to a sum over all six mass eigenfields. In this few-light scenario, the three independent flavor eigenfields ν_i must couple to at least three light-mass eigenfields ν_{m_j}.

5 Cosmology

5.1 Weighing Neutrinos

Any stable, two-component neutrino mass eigenfield ν_{m_j} that couples via U^*_{ij} to a flavor eigenfield must have a contemporary density of 115/cc due to thermal equilibrium in the early universe [15, 16]. Thus the present mass density of the eigenfield ν_{m_j} is $115\, m_j$/cc. The current critical density is $1.05 \times 10^4\, h^2\,\mathrm{eV/cc}$, where h is the Hubble parameter in units of 100 km/sec/Mpc. Thus if $h \approx 0.65$, then the contribution of ν_{m_j} to the neutrino critical density Ω_ν is $m_j/40\,\mathrm{eV}$. So the contribution to Ω_ν of the stable, light mass eigenfields ν_{m_j} that couple to the three flavor neutrinos is

$$\Omega_\nu = \left(\sum_{j \text{ light}} m_j \right) \frac{1}{40\,\mathrm{eV}}. \tag{56}$$

Since Ω_ν is the neutrino contribution to Hot Dark Matter, and since the HDM part of Ω_M is probably less than 0.2, we may conservatively conclude that $\Omega_\nu \lesssim 0.2$. Thus we arrive at an approximate upper bound [15, 16] for the sum of the masses m_j of the light, stable neutrinos that couple to the three flavor eigenfields:

$$\sum_{j \text{ light}} m_j \lesssim 8\,\mathrm{eV}. \tag{57}$$

Under other assumptions Fukugita, Liu, and Sugiyama [17] have derived limits on the sum of the light neutrino masses that are in the range of 2 to 5 eV. The Sloan Digital Sky Survey [18] will measure Ω_ν and weigh the light neutrinos [19]. Within the context of a minimally extended standard model, these cosmological bounds and the LEP constraint (54) imply that the Z gauge boson couples to between 3 and 6 very light neutrinos.

5.2 BBN Bounds on Light Sterile Neutrinos

It is open question whether active neutrinos oscillate mainly into other active neutrinos or mainly into sterile neutrinos. If they oscillate mainly into sterile neutrinos, then in the early universe electron neutrinos before they decoupled might have brought sterile neutrinos into chemical equilibrium [20], increasing the effective number $N_\nu^{\rm eff}$ of two-component neutrino species. A higher $N_\nu^{\rm eff}$ would have raised the energy density ρ of the universe and therefore its rate $\dot{R}$ of expansion since $\dot{R}^2 = (8\pi G/3)\rho R^2 - k$ where k is the curvature. The neutron-to-proton ratio would then have been higher at its freeze-out ($T \approx 0.7$ MeV), and the abundance Y of helium higher than is observed ($Y \approx 0.24 \pm 0.01$). In the case of maximal mixing between ν_μ and $\nu_{\mu s}$, the conservative BBN limit $N_\nu^{\rm eff} < 4$ would impose on $\delta m^2_{\nu_\mu} = |m^2_{\nu_\mu} - m^2_{\nu_{\mu s}}|$ an upper limit of the order of 10^{-6} eV^2 [20]. This big-bang-nucleosynthesis (BBN) constraint would rule out the possibility that the atmospheric muon neutrinos oscillate mainly into sterile muon neutrinos.

5.3 Lepton Asymmetries

However an excess of ν_e over $\bar{\nu}_e$, or *vice versa*, when the temperature was between 30 and 0.7 MeV, would have modified the reaction rates $n + \nu_e \leftrightarrow p + e^-$ and $n + e^+ \leftrightarrow p + \bar{\nu}_e$ and consequently changed the effective number of neutrino species $N_\nu^{\rm eff}$. Thus Foot and Volkas have pointed out [21] that the BBN constraints on active-sterile neutrino oscillations depend upon the assumption that the lepton-number asymmetry of the early universe was negligible. They also have argued [21] that active-sterile neutrino oscillations could themselves have generated a significant lepton-number asymmetry at the relevant temperatures. So the BBN limits on $N_\nu^{\rm eff}$ and $\delta m^2_{\nu_\mu}$ may not be valid. And the recent measurement of the cosmic microwave background (CMB) radiation by the Boomerang experiment has called into question the precision of BBN results; it now appears that the BBN bound on the baryon density ought to be raised by about 50% to $0.024 < h^2\Omega_b < 0.042$ [22].

The Affleck-Dine mechanism [25] has been suggested as a second way of generating B and L asymmetries in the minimal supersymmetric model [27] and its extensions [26].

5.4 Baryon Number of the Universe

K. Dick, M. Lindner, M. Ratz, and D. Wright [23] recently showed how a universe that conserves $B-L$ might have started with $B=L=0$ and still have arrived at the observed current baryon-to-photon ratio as long as the masses of the neutrinos are suitably small. In their model a density of $B+L$ produced at GUT temperatures is equilibrated between left- and right-handed (then massless) particles by Yukawa processes at temperatures above 1 TeV. Sphalerons [24] then washed out left-handed baryons and leptons, reducing $B+L$ while conserving $B-L$. If the Yukawa couplings of the neutrinos are so weak as to yield (purely Dirac) neutrino masses less than about 10 keV, then the equilibration of left- and right-handed neutrinos could not have kept up with the sphaleron washout, and an excess of left-handed baryons equal to the excess of right-handed neutrinos, which are immune to sphaleron washout, would have survived the electroweak era. This mechanism is both a potential explanation of the baryon number of the universe and a third way of generating lepton asymmetries.

Since $B-L$ is exactly conserved in this model, neutrino oscillations occur only among the three active flavors. But this model might work even if $B-L$ were slightly broken.

6 Oscillations

It follows from the action density (49) and from arguments presented elsewhere [29] that the lowest-order amplitude $A(\nu_i \to \nu_{i'})$ for a neutrino ν_i (*e.g.,* produced by a charged lepton e_i) after propagating with energy E a distance L as some light-mass eigenfield of mass $m_j \ll E$ to appear as a neutrino ν_i' (*e.g.,* producing a charged lepton e_i') is

$$A(\nu_i \to \nu_{i'}) = \sum_{j \text{ light}} U^*_{i'j} U_{ij} e^{-im_j^2 L/(2E)}. \tag{58}$$

In view of (46), the lowest-order amplitude for the anti-process, $\bar{\nu}_i \to \bar{\nu}_{i'}$, is

$$A(\bar{\nu}_i \to \bar{\nu}_{i'}) = \sum_{j \text{ light}} U_{i'j} U^*_{ij} e^{-im_j^2 L/(2E)}, \tag{59}$$

which is $A(\nu_i \to \nu_{i'})$ with the matrix elements of U replaced by their complex conjugates [28].

The corresponding probabilities to lowest order are

$$P(\nu_i \to \nu_{i'}) = \sum_{j,j' \,\text{light}} U^*_{i'j} U_{ij} U_{i'j'} U^*_{ij'} \exp\left[i(m^2_{j'} - m^2_j)L/(2E)\right] \tag{60}$$

and

$$P(\bar{\nu}_i \to \bar{\nu}_{i'}) = \sum_{j,j' \,\text{light}} U^*_{i'j} U_{ij} U_{i'j'} U^*_{ij'} \exp\left[i(m^2_j - m^2_{j'})L/(2E)\right] \tag{61}$$

both of which, if all six neutrinos are light, approach $\delta_{ii'}$ in the limit $L/E \to 0$.

A measure of CP violation is provided by the asymmetry parameter $A(i, i')$ defined as

$$A(i, i') = \frac{P(\nu_i \to \nu_{i'}) - P(\bar{\nu}_i \to \bar{\nu}_{i'})}{P(\nu_i \to \nu_{i'}) - P(\bar{\nu}_i \to \bar{\nu}_{i'})} \tag{62}$$

and given by

$$A(i, i') = \frac{\sum_{j,j' \,\text{light}} Im\left(U^*_{i'j} U_{ij} U_{i'j'} U^*_{ij'}\right) \sin\left[(m^2_j - m^2_{j'})L/(2E)\right]}{\sum_{j,j' \,\text{light}} Re\left(U^*_{i'j} U_{ij} U_{i'j'} U^*_{ij'}\right) \cos\left[(m^2_j - m^2_{j'})L/(2E)\right]}. \tag{63}$$

As we shall see in the next section, if the six neutrinos are light and either purely Dirac or purely Majorana, then for each $i = e, \mu, \tau$ the sum over i' is unity:

$$\sum_{i'=e}^{\tau} P(\nu_i \to \nu_{i'}) = 1; \tag{64}$$

but if the neutrinos are nearly but not quite Dirac fermions, then this sum of probabilities tends to be about

$$\sum_{i'=e}^{\tau} P(\nu_i \to \nu_{i'}) \approx \frac{1}{2}. \tag{65}$$

If for simplicity we stretch the error bars on the Chlorine experiment, then the solar neutrino experiments, especially Gallex and SAGE, show a diminution of electron neutrinos by a factor of about one-half:

$$P_{\text{sol}}(\nu_e \to \nu_e) \approx \frac{1}{2}, \tag{66}$$

which requires a pair of mass eigenstates whose squared masses differ by at least $\sim 10^{-10}\,\mathrm{eV}^2$ [15]. The reactor experiments, Palo Verde and especially Chooz, imply that these squared masses differ by less than $\sim 10^{-3}\,\mathrm{eV}^2$ [15].

The atmospheric neutrino experiments, Soudan II, Kamiokande III, IMB-3, and especially SuperKamiokande, show a diminution of muon neutrinos and antineutrinos by about one-third:

$$P_{\mathrm{atm}}(\nu_\mu \to \nu_\mu) \approx \frac{2}{3}, \tag{67}$$

which requires a pair of mass eigenstates whose squared masses differ by $10^{-3} \lesssim |m_j^2 - m_k^2| \lesssim 10^{-2}\,\mathrm{eV}^2$ [15].

If the LSND neutrino experiment is correct, then it requires a pair of states whose squared masses differ by $10^{-1} \lesssim |m_j^2 - m_k^2| \lesssim 10^{+1}\,\mathrm{eV}^2$ [15].

7 The $B - L$ Model

When the Majorana mass matrices E and F are both zero, the action density (23) is invariant under the $U(1)$ transformation

$$N' = e^{i\theta G}\,N \tag{68}$$

in which the 6×6 matrix G is the block-diagonal matrix

$$G = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{69}$$

with I the 3×3 identity matrix. The kinetic part of (23) is clearly invariant under this transformation. The mass terms are invariant only when the anti-commutator

$$\{\mathcal{M}, G\} = 2\begin{pmatrix} F & 0 \\ 0 & -E \end{pmatrix} = 0 \tag{70}$$

vanishes.

This $U(1)$ symmetry is the restriction to the neutrino sector of the symmetry generated by baryon-minus-lepton number, $B - L$, which is exactly conserved in the standard model. A minimally extended standard model with right-handed neutrino fields n_{ri} and a Dirac mass matrix D but with no Majorana mass matrices, $E = F = 0$, also conserves $B - L$. When $B - L$ is exactly conserved, *i.e.,* when $D \neq 0$ but $E = F = 0$, then the six neutrino masses m_j collapse into three pairs of degenerate masses which may be combined into three Dirac neutrinos.

Suppose this symmetry is slightly broken by the Majorana mass matrices E and F. Then for random mass matrices D, E, and F, the six neutrino masses m_j will form three pairs of nearly degenerate masses as long as the ratio

$$\sin^2 x_\nu = \frac{\mathrm{Tr}(E^\dagger E + F^\dagger F)}{\mathrm{Tr}(2D^\dagger D + E^\dagger E + F^\dagger F)} \tag{71}$$

is small. For a generic mass matrix $\mathcal{M}$, the parameter $\sin^2 x_\nu$ lies between the extremes

$$0 \leq \sin^2 x_\nu \leq 1 \tag{72}$$

and characterizes the kind of the neutrinos. The parameter $\sin^2 x_\nu$ is zero for purely Dirac neutrinos and unity for purely Majorana neutrinos.

Let us now recall 't Hooft's definition [8] of naturalness: It is natural to assume that a parameter is small if the theory becomes more symmetrical when the parameter vanishes. In this sense it is natural to assume that the parameter $\sin^2 x_\nu$ is small because the minimally extended standard model becomes more symmetrical, conserving $B - L$, when $\sin^2 x_\nu = 0$.

In Fig. 1 the six neutrino masses m_j are plotted for a set of mass matrices $\mathcal{M}$ that differ only in the parameter $\sin x_\nu$. Apart from $\sin x_\nu$, every other parameter of the mass matrices $\mathcal{M}$ is a complex number $z = x + iy$ in which x and y were chosen randomly and uniformly on the interval $[-1\,\mathrm{eV}, 1\,\mathrm{eV}]$. It is clear in the figure that when $\sin^2 x_\nu \approx 0$, the six neutrino masses m_j coalesce into three nearly degenerate pairs. Although the six masses of the neutrinos are in the eV range, they form three pairs with very tiny mass differences when $\sin^2 x_\nu \approx 0$.

Thus the very small mass differences required by the solar and atmospheric experiments are naturally explained by the assumption that the symmetry generated by $B - L$ is broken only slightly by the Majorana mass matrices E and F. This same assumption implies that neutrinos are very nearly Dirac fermions and hence explains the very stringent upper limits on

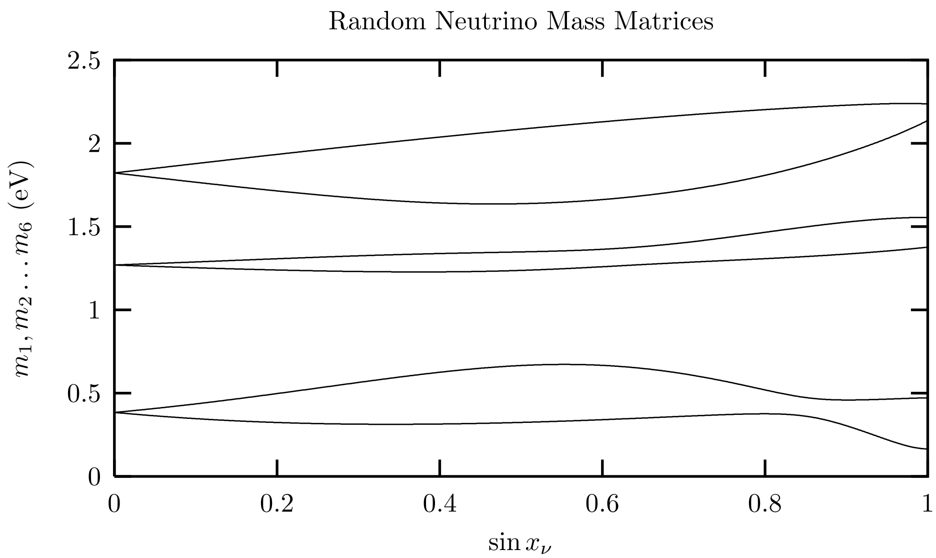

Figure 1: The six neutrino masses are plotted against the parameter $\sin x_\nu$ for a set of random 6×6 mass matrices.

neutrinoless double beta decay [15]. Because the masses of the six neutrinos may lie in the range of a few eV, instead of being squashed down to the meV range by the seesaw mechanism, they may contribute to hot dark matter in a way that is cosmologically significant. This $B - L$ model with $\sin^2 x_\nu \approx 0$ is the converse of the seesaw mechanism.

An Example

A simple illustration of these ideas is provided by the mass matrix $\mathcal{M}$

$$\mathcal{M} = \begin{pmatrix} f_1 \sin x & 0 & 0 & d_1 \cos x & 0 & 0 \\ 0 & f_2 \sin x & 0 & 0 & d_2 \cos x & 0 \\ 0 & 0 & f_3 \sin x & 0 & 0 & d_3 \cos x \\ d_1 \cos x & 0 & 0 & e_4 \sin x & 0 & 0 \\ 0 & d_2 \cos x & 0 & 0 & e_5 \sin x & 0 \\ 0 & 0 & d_3 \cos x & 0 & 0 & e_6 \sin x \end{pmatrix}, \quad (73)$$

in which all the elements are taken to be real and non-negative. At $x = \pi/2$, the Majorana parameter $\sin x_\nu$ is unity, and the neutrinos are purely

Majorana. The six flavor eigenfields $\nu_1, \nu_2, \nu_3, n_1, n_2, n_3$ are then also mass eigenfields with masses $f_1, f_2, f_3, e_1, e_2, e_3$, respectively.

For x between $\pi/2$ and zero, the fields ν_i and n_i for each $i = 1, 2, 3$ mix to form three pairs of mass eigenfields with masses

$$m_{i\pm} = \frac{1}{2} \left| (e_i + f_i) \sin x \pm \sqrt{(e_i - f_i)^2 \sin^2 x + 4d_i^2 \cos^2 x} \right|, \tag{74}$$

which are the singular values of $\mathcal{M}$. The difference of the squared masses of the ith pair is proportional to $\sin x$

$$m_{i+}^2 - m_{i-}^2 = (e_i + f_i) \sqrt{(e_i - f_i)^2 \sin^2 x + 4d_i^2 \cos^2 x} \, \sin x. \tag{75}$$

If the mass parameters, f_i, e_i, d_i, are comparable in magnitude, then the Majorana parameter $\sin x_\nu$ is roughly $\sin x$.

At $x = 0$, $x_\nu = 0$, the neutrinos are purely Dirac fermions, and the pair of mass eigenfields

$$\nu_{m_i+} = \frac{1}{\sqrt{2}} (\nu_i + n_i) \quad \text{and} \quad \nu_{m_i-} = \frac{-i}{\sqrt{2}} (\nu_i - n_i) \tag{76}$$

have the same mass or singular value, $m_i = |m_{i\pm}| = d_i$. Since these fields are degenerate, the linear combinations

$$\nu_i = \frac{1}{\sqrt{2}} (\nu_{m_i+} + i\nu_{m_i-}) \quad \text{and} \quad n_i = \frac{1}{\sqrt{2}} (\nu_{m_i+} - i\nu_{m_i-}) \tag{77}$$

are also mass eigenfields; they may be combined to form a single Dirac neutrino

$$\psi_i = \begin{pmatrix} \nu_i \\ i\sigma_2 n_i^* \end{pmatrix} \tag{78}$$

of mass m_i for each $i = 1, 2, 3$. It follows from Eqs. (31) and (33) that ψ_i satisfies Dirac's equation $(\gamma^n \partial_n + m_i)\psi_i = 0$. When x_ν is not zero or near zero, then there is no reason why any two of the masses m_i should be equal, and one may not combine two of the fields to form a Dirac neutrino.

For very small values of the angle x and of the parameter $\sin^2 x_\nu$, the neutrinos are nearly Dirac, and the three squared-mass differences

$$m_{i+}^2 - m_{i-}^2 \approx 2d_i(e_i + f_i) \cos x \sin x \tag{79}$$

being proportional to $\sin x \approx \sin x_\nu$, are very small, and the mass eigenfields are approximately

$$\nu_{m_i \pm} \approx \frac{1}{\sqrt{2}} (\nu_i \pm n_i) + \frac{e_i - f_i}{2\sqrt{2}\, d_i} \tan x \, n_i. \tag{80}$$

If $\sin^2 x_\nu = 0$, then there are three purely Dirac neutrinos, and the mixing matrix V is block diagonal

$$V = \begin{pmatrix} u^* & 0 \\ 0 & v \end{pmatrix} \tag{81}$$

in which the 3×3 unitary matrices u and v occur in the singular-value decomposition of the 3×3 matrix $D = u\, m\, v^\dagger$. If these three Dirac neutrinos are also light, then unitarity implies that the sum of the normalized probabilities is unity

$$\sum_{i'=e}^{\tau} P(\nu_i \to \nu_{i'}) = 1. \tag{82}$$

If $\sin^2 x_\nu = 1$, then this sum is also unity by unitarity because in this case the mixing matrix for the six purely Majorana neutrinos is

$$V = \begin{pmatrix} v_F & 0 \\ 0 & v_E \end{pmatrix}. \tag{83}$$

But if there are six light, nearly Dirac neutrinos, then each neutrino flavor ν_i will oscillate both into other neutrino flavor eigenfields and into sterile neutrino eigenfields. In this case this sum tends to be about a half

$$\sum_{i'=e}^{\tau} P(\nu_i \to \nu_{i'}) \approx \frac{1}{2}. \tag{84}$$

Fitting the Solar and Atmospheric Data

This intuition is supported by Figs. 2 and 3 which respectively display for 10000 random mass matrices $\mathcal{M}$ the sums of probabilities $\sum_{i'=e}^{\tau} P_{\rm sol}(\nu_e \to \nu_{i'})$ and $\sum_{i'=e}^{\tau} P_{\rm atm}(\nu_\mu \to \nu_{i'})$ as a function of the parameter $\sin x_\nu$. The plots clearly show that these sums tend to cluster around the value $\frac{1}{2}$ when $\sin x_\nu$ is small but not infinitesimal. The points at $\sin x_\nu = 0$ and at $\sin x_\nu = 1$

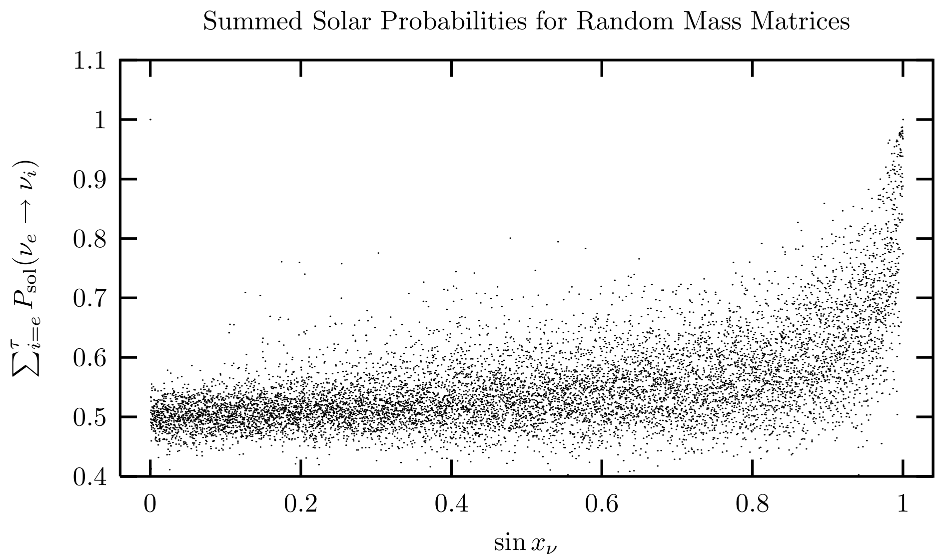

Figure 2: The sum of the probabilities $P_{\rm sol}(\nu_e \to \nu_i)$ for solar neutrinos summed over i for $i = e, \mu, \tau$ for 10000 random mass matrices $\mathcal{M}$ are plotted as a function of the parameter $\sin x_\nu$.

display the unitarity relation (82) for respectively purely Dirac and purely Majorana neutrinos.

In these scatter plots and in those that follow, every parameter of the 10000 matrices is a complex number $z = x + iy$ with x and y chosen randomly and uniformly from the interval $[-1\text{eV}, 1\text{eV}]$. The solar neutrinos are taken to have an energy of 1 MeV, and the probability (60) is averaged over one revolution of the Earth about the Sun. The atmospheric neutrinos are averaged over the atmosphere and over energies in the range of 1–30 GeV weighted by the flux of atmospheric muon neutrinos as a function of energy and local zenith angle given by the Bartol group in Table V of ref. [30].

If we continue to interpret the Chlorine experiment very conservatively, then the data of the solar and atmospheric experiments (66) and (67) require that

$$P_{\rm sol}(\nu_e \to \nu_e) \approx \frac{1}{2} \qquad \text{and} \qquad P_{\rm atm}(\nu_\mu \to \nu_\mu) \approx \frac{2}{3}. \tag{85}$$

But because of the approximate, empirical sum rule (84) for $i = e$ and μ, the only way in which the probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ can fit

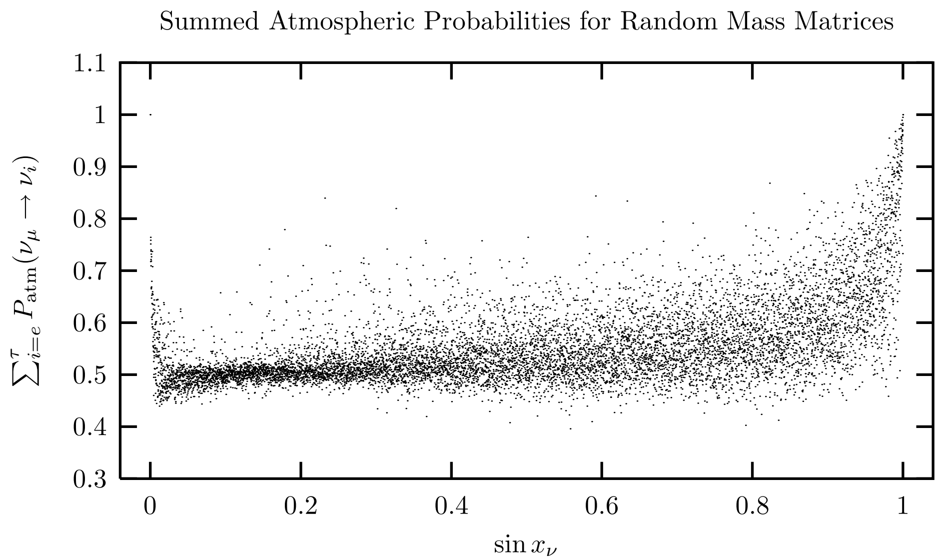

Figure 3: The sum of the probabilities $P_{\rm atm}(\nu_\mu \to \nu_i)$ for atmospheric neutrinos summed over i for $i = e, \mu, \tau$ for 10000 random mass matrices $\mathcal{M}$ are plotted as a function of the parameter $\sin x_\nu$.

these experimental results is if inter-generational mixing is suppressed so that ν_e oscillates into n_e and so that ν_μ oscillates into n_μ. In other words, random mass matrices $\mathcal{M}$, even with $\sin x_\nu \approx 0$, produce probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ (suitably averaged respectively over the Earth's orbit and over the atmosphere) that are too small. The reason is that for small $\sin x_\nu$ the flavor eigenfield ν_e or ν_μ necessarily is split into two mass eigenfields, which reduces the probabilities to $\sim \frac{1}{2}$; and so, if there is further inter-generational mixing, then these probabilities tend to be too small. This effect is illustrated in Fig. 4 in which the probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ are plotted for 10000 † random mass matrices $\mathcal{M}$ all with $\sin x_\nu = 0.003$. Figure 4 makes it clear that the probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ are too small to fit the experimental results (66) and (67).

The probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ tend to be somewhat larger when inter-generational mixing is limited. In Fig. 5 these probabilities are displayed for 10000 random mass matrices with $\sin x_\nu = 0.003$ and with

†Due to the arXiv limit of 650 kb on the length of articles, in some of the figures only 6000 points are plotted.

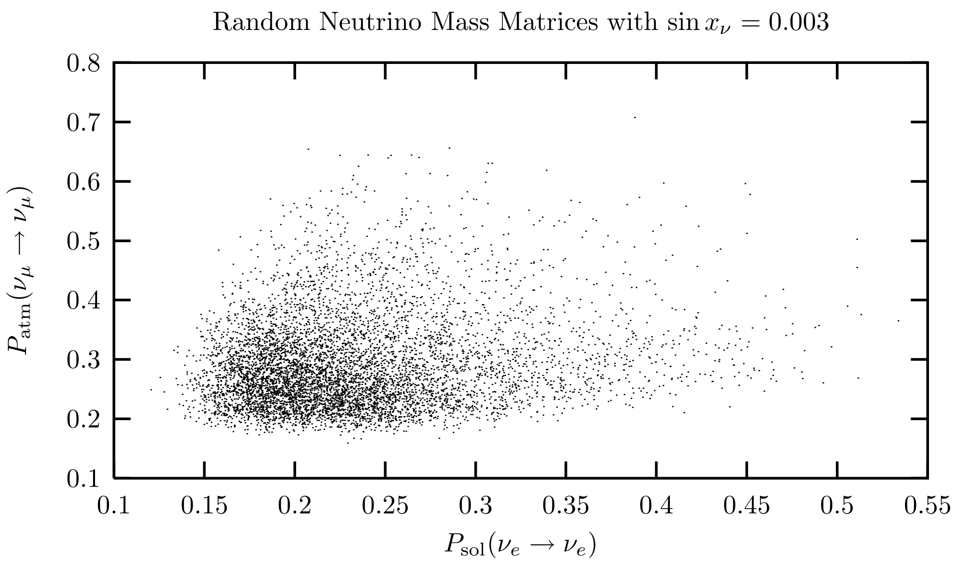

Figure 4: The probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$.

the singly off-diagonal matrix elements of D, E, and F suppressed by 0.2 and the doubly off-diagonal matrix elements suppressed by 0.004. The points in Fig. 5 are in much better agreement with (85) than are those of Fig. 4, but they still fail to match the experiments. Yet if the suppression of mixing is more severe, with factors respectively of 0.05 and 0.0025, then as shown in Fig. 6 the probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ do tend to cluster around $(\frac{1}{2}, \frac{2}{3})$ as required by the data.

It is possible to relax the factors that suppress inter-generational mixing back to 0.2 and 0.04 and improve the agreement with the experimental constraints (66) and (67) (while satisfying the CHOOZ constraint) provided that one also requires that there be a quark-like mass hierarchy. The points in Fig. 7 were generated from 10000 random mass matrices $\mathcal{M}$ with $\sin x_\nu = 0.003$ and CKM-suppression factors of 0.2 and 0.04 as in Fig. 5 and with the i, j-th elements of the mass matrices E, F, and D scaled by the factors $f(i) * f(j)$ where $\vec{f} = (0.2, 1, 2)$. Thus the mass matrix $\mathcal{M}$ has the τ, τ elements that are larger than its μ, μ elements and μ, μ elements that in turn are larger than its e, e elements. The clustering of the probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ around $(\frac{1}{2}, \frac{2}{3})$ in Fig. 7 shows that the ex-

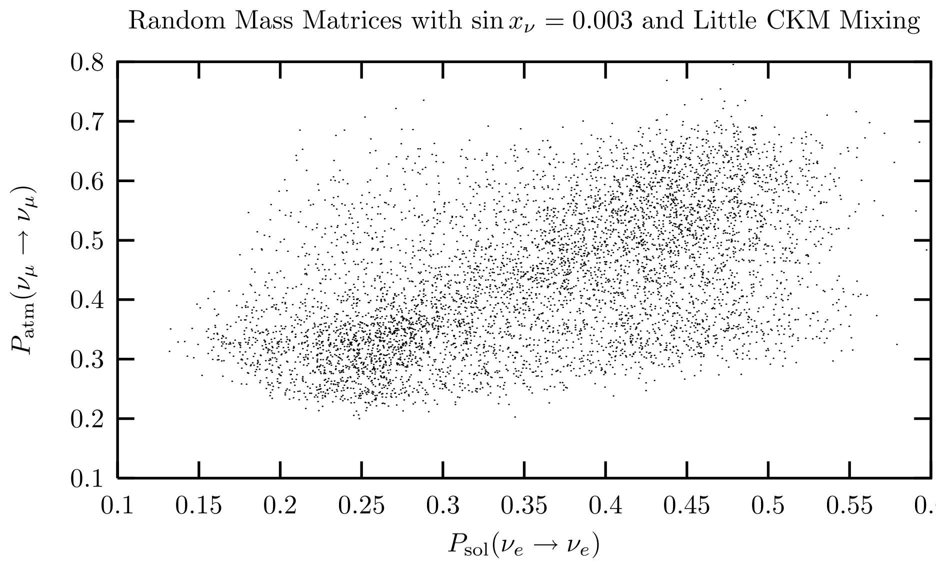

Figure 5: The probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$ and with inter-generational mixing suppressed by factors of 0.2.

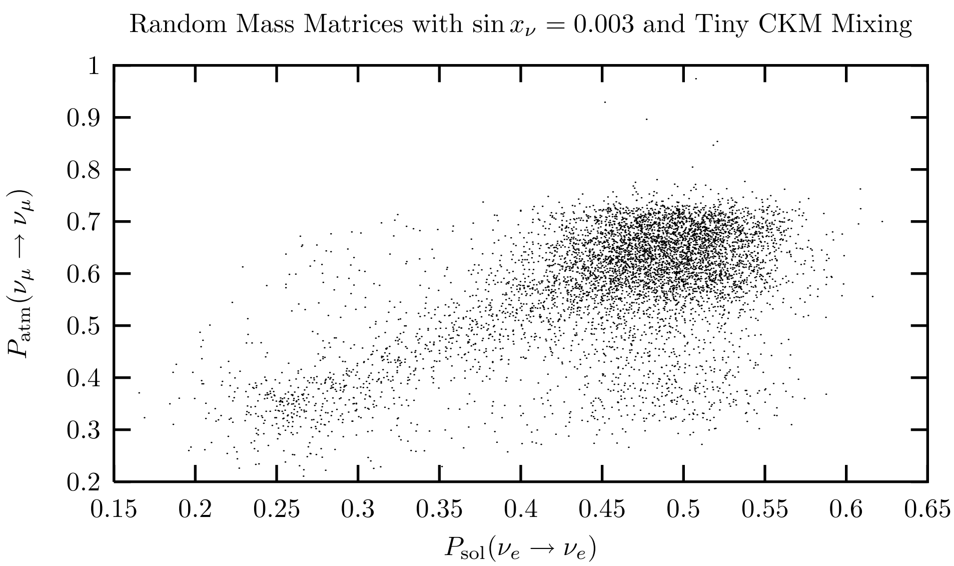

Figure 6: The probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$ and with inter-generational mixing suppressed by factors of 0.05.

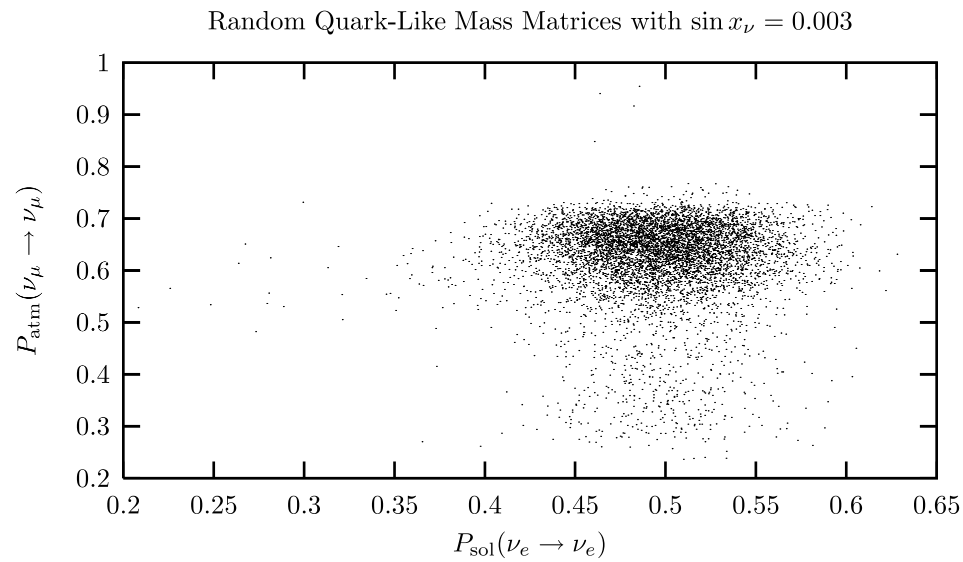

Figure 7: The probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$, with intergenerational mixing suppressed by factors of 0.2, and with a quark-like mass hierarchy.

perimental results (66) and (67) are satisfied. The vector $\vec{f}$ was tuned so as to nearly saturate the cosmological upper bound (57) of about 8 eV.

LSND, KARMEN2, and MiniBooNE

Because x_ν is small, and because neutrinos oscillate mainly into sterile neutrinos, the probabilities of the appearance of neutrinos are small, as shown by LSND and by KARMEN2. But since the mass differences among the three nearly degenerate pairs of neutrinos can lie in the eV range, such oscillations should be observable. In Fig. 8 for a set of 10000 mass matrices generated randomly with the same parameters as for Fig. 7, the probabilities $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ are shown for LSND and KARMEN2.

In Fig. 9 for a set of 10000 mass matrices generated randomly with the same parameters as for Figs. 7 and 8, the probabilities $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_\mu)$ for MiniBooNE are plotted. If intergenerational mixing is sup-

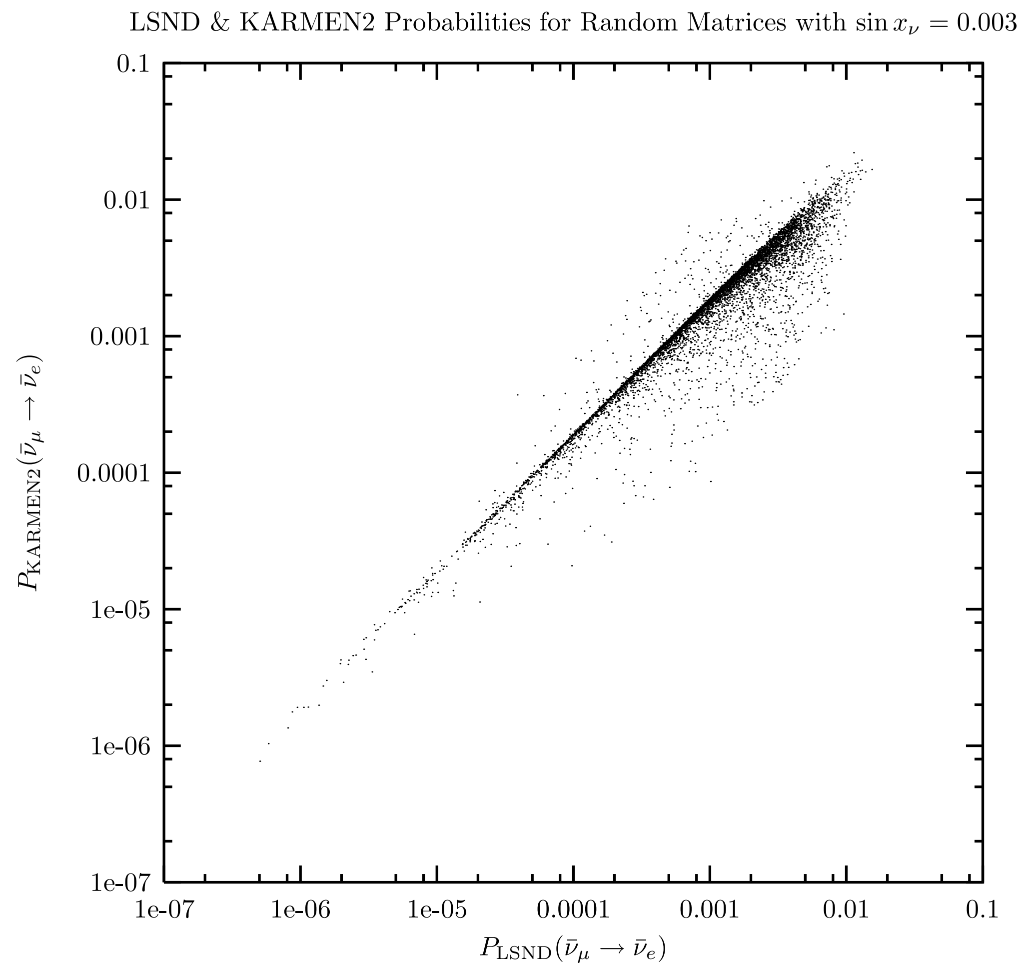

Figure 8: The probabilities $P_{\mathrm{LSND}}(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P_{\mathrm{KARMEN2}}(\bar{\nu}_\mu \to \bar{\nu}_e)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$, with inter-generational mixing suppressed by factors of 0.2, and with a quark-like mass hierarchy.

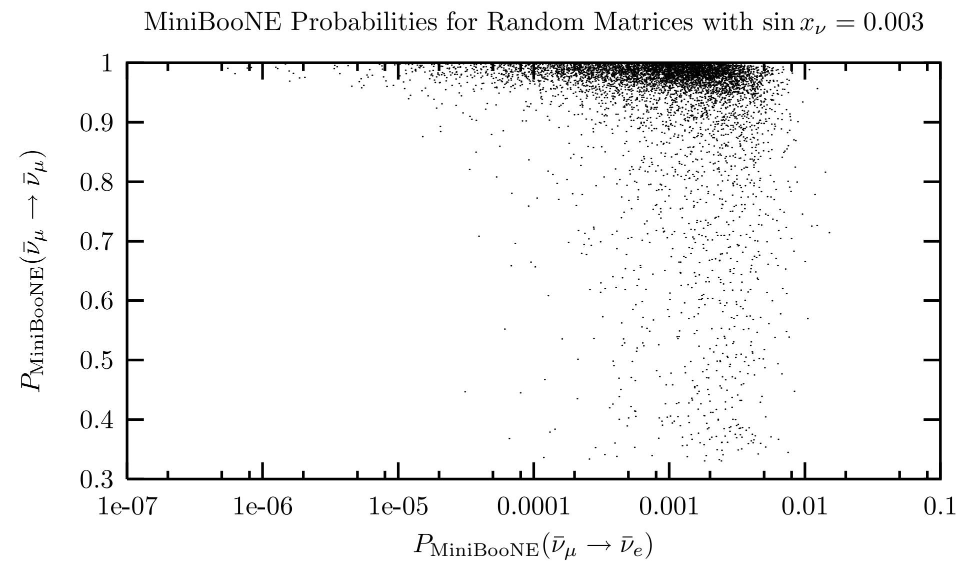

Figure 9: The probabilities $P_{\mathrm{MiniBooNE}}(\bar\nu_\mu \to \bar\nu_e)$ and $P_{\mathrm{MiniBooNE}}(\bar\nu_\mu \to \bar\nu_\mu)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$, with inter-generational mixing suppressed by factors of 0.2, and with a quark-like mass hierarchy.

pressed only by factors of 0.2 and if MiniBooNE can achieve a sensitivity of 0.001 for $\bar\nu_\mu \to \bar\nu_e$ and a precision of 0.01 for $\bar\nu_\mu \to \bar\nu_\mu$, then it has a good chance of seeing both the appearance of $\bar\nu_e$ and the disappearance of $\bar\nu_\mu$.

It may be, however, that intergenerational mixing is suppressed only by factors of 0.1. For as shown in Fig. 10, a set of 10000 matrices randomly generated, with CKM suppression factors of 0.1 for the off-diagonal terms and 0.01 for the off-off-diagonal ones and with the mass-hierarchy vector $\vec{f} = (.2, 1, 2)$, agrees with the solar and atmospheric experiments just as well as those of Fig. 7. With these parameters the probabilities $P(\bar\nu_\mu \to \bar\nu_e)$ at LSND and KARMEN2 are smaller, as shown in Fig. 11. The chances of detection at MiniBooNE are similarly reduced as shown in Fig. 12.

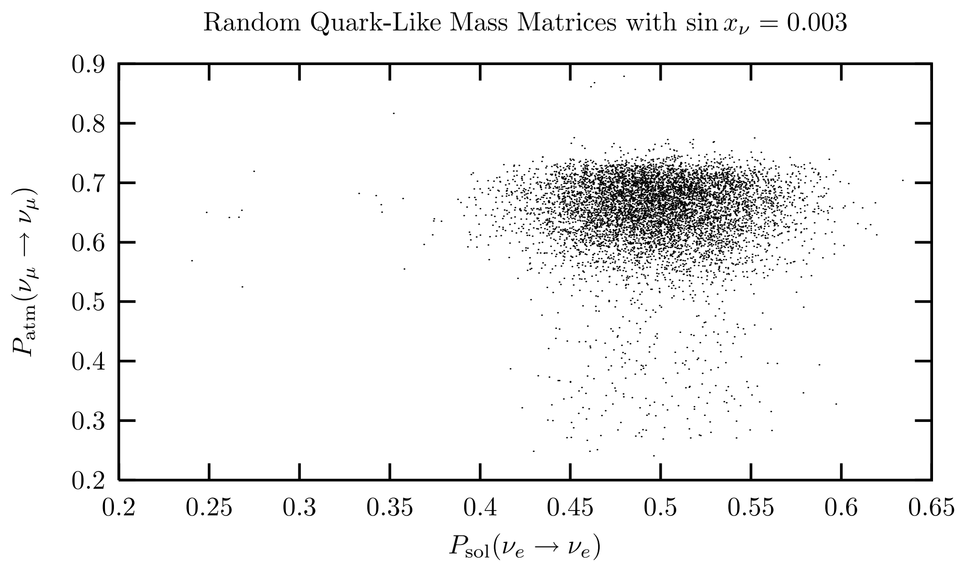

Figure 10: The probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$, with inter-generational mixing suppressed by factors of 0.1, and with a quark-like mass hierarchy.

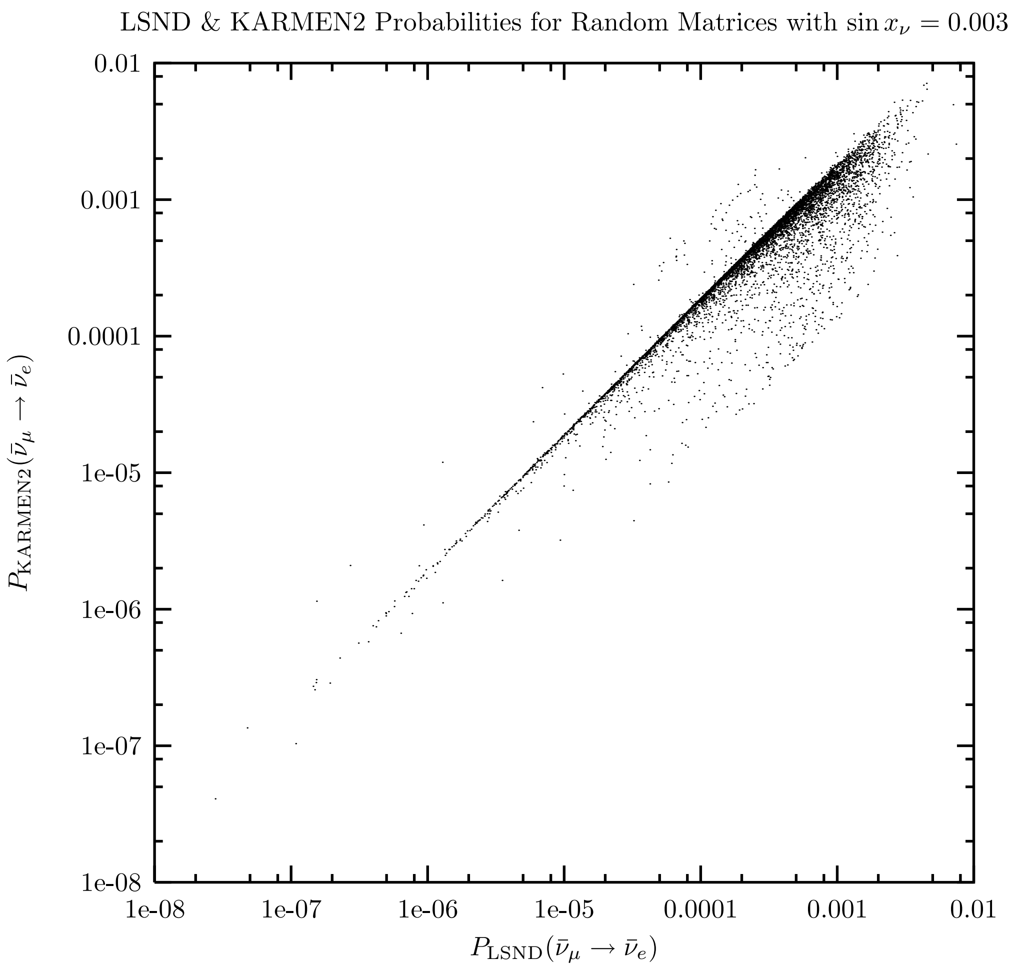

Figure 11: The probabilities $P_{\mathrm{LSND}}(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P_{\mathrm{KARMEN2}}(\bar{\nu}_\mu \to \bar{\nu}_e)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$, with inter-generational mixing suppressed by factors of 0.1, and with a quark-like mass hierarchy.

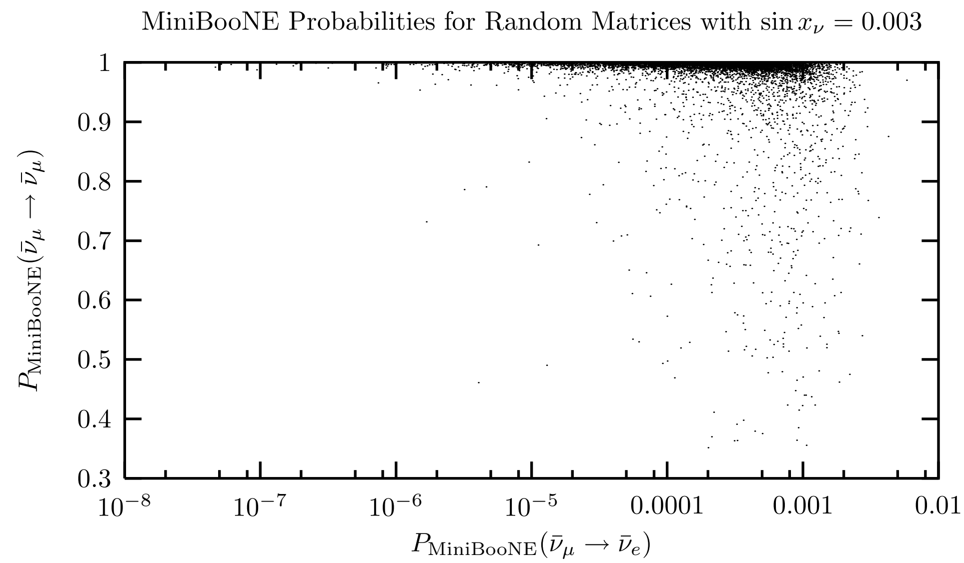

Figure 12: The probabilities $P_{\rm MiniBooNE}(\bar\nu_\mu \to \bar\nu_e)$ and $P_{\rm MiniBooNE}(\bar\nu_\mu \to \bar\nu_\mu)$ for 10000 random mass matrices $\mathcal{M}$ all with the parameter $\sin x_\nu = 0.003$, with inter-generational mixing suppressed by factors of 0.1, and with a quark-like mass hierarchy.

Neutrinoless Double Beta Decay

Neutrinoless double beta decay occurs when a right-handed antineutrino emitted in one decay $n \to p + e^- + \bar{\nu}_e$ is absorbed as a left-handed neutrino in another decay $\nu_e + n \to p + e^-$. To lowest order these decays proceed via the Majorana mass term $-iF^*_{ee}\nu_e^\dagger\sigma_2\nu_e^\dagger$. Let us introduce a second angle y_ν defined by

$$\sin^2 y_\nu = \frac{\mathrm{Tr}(F^\dagger F)}{\mathrm{Tr}(E^\dagger E + F^\dagger F)}. \tag{86}$$

We have seen that we may fit the experimental data (66) and (67) by assuming that $\sin x_\nu \simeq 0.003$ and by requiring the mass matrices E, F, and D to exhibit quark-like mass hierarchies with little inter-generational mixing. Under these conditions the rate of $0\nu\beta\beta$ decay is limited by the factor

$$|F_{ee}|^2 \lesssim \sin^2 x_\nu \, \sin^2 y_\nu \, m^2_{\nu_e}, \tag{87}$$

in which m_{ν_e} is the heavier of the lightest two neutrino masses. Thus the rate of $0\nu\beta\beta$ decay is suppressed by an extra factor $\sim \sin^2 x_\nu \sin^2 y_\nu \lesssim 10^{-5}$ resulting in lifetimes $T_{\frac{1}{2},0\nu\beta\beta} > 2\times10^{27}$ yr. The $B-L$ model therefore explains why neutrinoless double beta decay has not been seen and predicts that the current and upcoming experiments Heidelberg/Moscow, IGEX, GENIUS, and CUORE will not see $0\nu\beta\beta$ decay.

8 Conclusions

The standard model slightly extended to include right-handed neutrino fields exactly conserves $B - L$ if all Majorana mass terms vanish. It is therefore natural [8] to assume that the Majorana mass terms are small compared to the Dirac mass terms. An angle x_ν is introduced that characterizes the relative importance of these two kinds of mass terms. When this parameter is very small, then the neutrinos are nearly Dirac and only slightly Majorana. In this case the six neutrino masses m_j coalesce into three pairs of nearly degenerate masses. Thus the very tiny mass differences seen in the solar and atmospheric neutrino experiments are simply explained by the natural assumption that $x_\nu \approx 0$ or equivalently that $B - L$ is almost conserved.

In these experiments the probabilities $P_{\rm sol}(\nu_e \to \nu_e)$ and $P_{\rm atm}(\nu_\mu \to \nu_\mu)$ are respectively approximately one half and two thirds. One may fit these probabilities with random mass matrices in the eV range by setting $\sin x_\nu = 0.003$ and requiring the neutrino mass matrices E, F, and D to exhibit quark-like mass hierarchies with little inter-generational mixing.

The three mass differences among the three nearly degenerate pairs are constrained only by the cosmological bound $\sum_j m_j \lesssim 8\,\text{eV}$ and may lie in the range $[0.01, 4]\,\text{eV}$.

This $B - L$ model leads to these predictions:

1. Because $\sin^2 x_\nu \approx 0$ and because inter-generational mixing is suppressed, neutrinos oscillate mainly into sterile neutrinos of the same flavor and not into neutrinos of other flavors. Hence rates for the appearance of neutrinos, $P(\nu_i \to \nu_{i'})$ with $i \neq i'$, are very low, as shown by LSND and by KARMEN2. But because the mass differences among the three nearly degenerate pairs of masses can lie in the eV range, MiniBooNE may detect $\bar{\nu}_\mu \to \bar{\nu}_e$.

2. The assumption that $\sin^2 x_\nu$ is very small naturally explains the very small differences of squared masses seen in the solar and atmospheric experiments without requiring that the neutrino masses themselves be very small. Thus the neutrinos may very well saturate the cosmological bound, $\sum_j m_j \lesssim 8\,\text{eV}$. In fact the masses associated with the points of Figs. 7–12 do nearly saturate this bound. Neutrinos thus may well be an important part of hot dark matter.

3. The disappearance of ν_τ should *in principle* be observable.

4. In the $B - L$ model, the rate of neutrinoless double beta decay is suppressed by an extra factor $\sim \sin^2 x_\nu \, \sin^2 y_\nu \lesssim 10^{-5}$ resulting in lifetimes greater than $2 \times 10^{27}\,\text{yr}$. Thus the current and upcoming experiments Heidelberg/Moscow, IGEX, GENIUS, and CUORE will not see $0\nu\beta\beta$ decay.

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Appendix: LAPACK

The thousands of 6×6 matrix computations displayed in the several graphs were performed in the inexpensive Lintel computing environment of Intel chips, Red Hat Linux, the Portland Group's Fortran compiler, and the linear-algebra software of the LAPACK collaboration. This appendix describes how to use LAPACK to perform the singular-value decomposition of an arbitrary 6x6 complex matrix and the Takagi factorization of a 6x6 complex symmetric matrix.

The driver subroutine ZGESVD [10] does a singular-value decomposition of an arbitrary matrix A. The FORTRAN 90 call is

```
call ZGESVD( JOBU, JOBVT, M, N, A, LDA, S, U, LDU, &
VT, LDVT, WORK, LWORK, RWORK, INFO )
```

in which JOBU = 'A', JOBVT = 'A', M = N = LDA = LDVT = 6, WORK is an LWORK-dimensional double-complex vector, and RWORK is a 26-dimensional double-precision real vector. ZGESVD performs a singular-value decomposition of the 6×6 double-complex matrix A which is the mass matrix

$\mathcal{M}$ and reports its singular values, which are the neutrino masses m_j, as the 6-dimensional double-precision real vector S. The matrices U and $V^\dagger$ are contained in the 6×6 double-complex matrices U and VT. The integer INFO describes the level of success of the computation. The best value of the system-dependent integer LWORK is reported as the real part of WORK(1); for the computations of this paper, which were done on Pentium II's and III's, LWORK was 396. If JOBU is 'A' and JOBVT is 'A', then the subroutine call destroys the matrix A. Thus one must redefine A before every call to ZGESVD.

If the matrix A is symmetric, then one may convert the singular-value decomposition $A = USV^\dagger$ into the Takagi factorization $A = WSW^\top$ by the FORTRAN 90 equivalent of

$$W_{ij} = U_{ij} \sqrt{\frac{V^\dagger_{jk}}{|V^\dagger_{jk}|} \frac{|U_{kj}|}{U_{kj}}} \tag{88}$$

where the index k is chosen to avoid any possible singularity.

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[10] The subroutine ZGESVD and all the subroutines and functions on which it depends are freely available from www.netlib.org/lapack/ and come prebuilt with the Portland Group's FORTRAN90 compiler which may be downloaded from www.pgroup.com/.

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