The Fourth Root of Gravity*

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Abstract

By attaching basis vectors to the components of matter fields, one may render free action densities fully covariant. Both the connection and the tetrads are quadratic forms in these basis vectors. The metric of spacetime, which is quadratic in the tetrads, is then quartic in the basis vectors.

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Introduction

We are accustomed to writing matter fields as column vectors with components ψ_a. For a two-component right-handed spinor R, we might write

$$R(x) = \begin{pmatrix} R_1(x) \\ R_2(x) \end{pmatrix} \tag{1}$$

in a kind of mathematical baby talk. What we should write is

$$R(x) = R_a(x)e^a(x) \tag{2}$$

in which the complex basis vectors $e^a(x)$ for $a = 1, 2$ depend upon the space-time point x. It is not obvious how many components $e^a_r(x)$ these vectors ought to possess; in this essay they will have two complex components. We shall see that by attaching these vectors $e^a(x)$ to the components $R_a(x)$, we may render the free Dirac action density fully covariant.

The Effect of the Vectors $e^a(x)$

Let us consider the free action density S for the spinor R,

$$S = iR(x)^\dagger \sigma^\mu \partial_\mu R(x), \tag{3}$$

where $\sigma^\mu = (1, \vec{\sigma})$ are the Pauli matrices. If we write R with its vectors e^a, then we find

$$S = iR_c(x)^\dagger e^{c\dagger}_r(x) \sigma^\mu_{rs} \left[e^a_s(x) R_{a,\mu}(x) + R_a(x) e^a_{s,\mu}(x) \right]. \tag{4}$$

By inserting the two-by-two identity matrix

$$1 = e^a(x)e_a(x)^\dagger \tag{5}$$

before the term $R_a(x)e^a_{s,\mu}(x)$ and by renaming some indices, we may write the action density in the form

$$S = iR_c(x)^\dagger e^{c\dagger}(x)\sigma^\mu \left[e^a(x)R_{a,\mu}(x) + R_b(x)e^a(x)e_a(x)^\dagger \cdot e^b_{,\mu}(x)\right]. \tag{6}$$

Thus if we identify the connection $A_\mu(x)$ as

$$A^b_{\mu a}(x) = e_a(x)^\dagger \cdot e^b_{,\mu}(x) \tag{7}$$

and the tetrad $V^\mu_i(x)$ by the relation

$$V^\mu_i(x)\sigma^{ica} = e^{c\dagger}_k(x)\sigma^\mu_{kl}e^a_l(x), \tag{8}$$

or equivalently as the trace

$$V^\mu_i(x) = (1/2)\mathrm{tr}(\sigma^i e^\dagger \sigma^\mu e), \tag{9}$$

then we find for the action density the usual expression

$$S = iR_c(x)^\dagger V^\mu_i(x)\sigma^{ica}D^b_{\mu a}R_b(x) \tag{10}$$

with

$$D^b_{\mu a}R_b(x) = \left[\delta^b_a\partial_\mu + A^b_{\mu a}(x)\right]R_b(x). \tag{11}$$

A similar massage performed upon the action density

$$iL^\dagger\sigma_\mu\partial_\mu L \tag{12}$$

of the left-handed spinor $L(x) = L_a(x) f_a(x)$ yields a left-handed tetrad $W^\nu_j(x)$. In terms of these tetrads the metric of spacetime is [1]

$$g^{\mu\nu}(x) = (1/2)(V^\mu_i W^\nu_i + V^\nu_i W^\mu_i). \tag{13}$$

Since the metric is quartic in the vectors e^a and f^b, these basis vectors may be thought of as fourth roots of the metric. The present formalism expresses both the metric $g_{\mu\nu}$ and the connection $A^b_{\mu a}$ in terms of the vectors e^a and f^b, which in turn are parts of the vectors R and L.

A Minimal Theory of Gravitation

As the action for the metric $g_{\mu\nu}$ itself, one may choose among various actions constructed from the vectors e^a and f^b and from the connection $A^b_{\mu a}$. The simplest and most conservative choice is to use metric $g_{\mu\nu}$ to build the usual Einstein action density

$$S_g = \sqrt{-g(x)}\, R(x) \tag{14}$$

and to require the matrices e and f to have determinant unity and to obey the relation

$$e = f^{-1\dagger}. \tag{15}$$

The two tetrads V and W are then identical, and the gravitational field is described by three complex numbers at each point x of spacetime.

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References

[1] K. Cahill, *Phys. Rev.* D26 (1982) 1916.