and the functional delta function

$$\delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}] = \prod_x \delta(\boldsymbol{\nabla} \cdot \boldsymbol{A}(x)) \tag{16.163}$$

enforces the Coulomb-gauge condition. The term $\mathcal{L}_m$ is the action density of the matter field ψ.

Tricks are available. We introduce a new field $A^0(x)$ and consider the factor

$$F = \int \exp\left[i \int \frac{1}{2} \left(\boldsymbol{\nabla} A^0 + \boldsymbol{\nabla} \triangle^{-1} j^0 \right)^2 d^4x \right] DA^0 \tag{16.164}$$

which is just a *number* independent of the charge density j^0 since we can cancel the j^0 term by shifting A^0. By $\triangle^{-1}$, we mean $-1/4\pi|\boldsymbol{x} - \boldsymbol{y}|$. By integrating by parts, we can write the number F as (exercise 16.21)

$$\begin{aligned} F &= \int \exp\left[i \int \tfrac{1}{2} \left(\boldsymbol{\nabla} A^0 \right)^2 - A^0 j^0 - \tfrac{1}{2} j^0 \triangle^{-1} j^0 \; d^4x \right] DA^0 \\ &= \int \exp\left[i \int \tfrac{1}{2} \left(\boldsymbol{\nabla} A^0 \right)^2 - A^0 j^0 \; d^4x + i \int V_C \, dt \right] DA^0. \end{aligned} \tag{16.165}$$

So when we multiply the numerator and denominator of the amplitude (16.161) by F, the awkward Coulomb term cancels, and we get

$$\langle \Omega | \mathcal{T} \left[\mathcal{O}_1 \dots \mathcal{O}_n \right] | \Omega \rangle = \frac{\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS'} \, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}] \, DA \, D\psi}{\int e^{iS'} \, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}] \, DA \, D\psi} \tag{16.166}$$

where now DA includes all four components A^μ and

$$S' = \int \tfrac{1}{2} \dot{\boldsymbol{A}}^2 - \tfrac{1}{2} (\boldsymbol{\nabla} \times \boldsymbol{A})^2 + \tfrac{1}{2} \left(\boldsymbol{\nabla} A^0 \right)^2 + \boldsymbol{A} \cdot \boldsymbol{j} - A^0 j^0 + \mathcal{L}_m \, d^4x. \tag{16.167}$$

Since the delta-function $\delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}]$ enforces the Coulomb-gauge condition, we can add to the action S' the term $(\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}}) \, A^0$ which is $-\dot{\boldsymbol{A}} \cdot \boldsymbol{\nabla} A^0$ after we integrate by parts and drop the surface term. This extra term makes the action gauge invariant

$$\begin{aligned} S &= \int \tfrac{1}{2} (\dot{\boldsymbol{A}} - \boldsymbol{\nabla} \boldsymbol{A^0})^2 - \tfrac{1}{2} (\boldsymbol{\nabla} \times \boldsymbol{A})^2 + \boldsymbol{A} \cdot \boldsymbol{j} - A^0 j^0 + \mathcal{L}_m \, d^4x \\ &= \int - \tfrac{1}{4} F_{ab} \, F^{ab} + A^b j_b + \mathcal{L}_m \; d^4x. \end{aligned} \tag{16.168}$$