and is positive for all functions $h(t)$. The stationary classical trajectory

$$q(t) = \frac{t - t_1}{t_2 - t_1}\, q(t_2) + \frac{t_2 - t}{t_2 - t_1}\, q(t_1) \tag{15.24}$$

is a **minimum** of the action $S_0[q]$. □

The second functional derivative of the action $S[q]$ (15.2) is

$$\begin{aligned} \delta^2 S[q][h] &= \frac{d^2}{d\epsilon^2} \int_{t_1}^{t_2} dt \left[\frac{m}{2} \left(\frac{dq(t)}{dt} + \epsilon \frac{dh(t)}{dt} \right)^2 - V(q(t) + \epsilon h(t)) \right] \Bigg|_{\epsilon=0} \\ &= \int_{t_1}^{t_2} dt \left[m \left(\frac{dh(t)}{dt} \right)^2 - \frac{\partial^2 V(q(t))}{\partial q^2(t)} h^2(t) \right] \end{aligned} \tag{15.25}$$

and it can be positive, zero, or negative. Chaos sometimes arises in systems of several particles when the second variation of $S[q]$ about a stationary path is negative, $\delta^2 S[q][h] < 0$ while $\delta S[q][h] = 0$.

The nth functional derivative is defined as

$$\delta^n G[f][h] = \frac{d^n}{d\epsilon^n}\, G[f + \epsilon h]\big|_{\epsilon=0}\,. \tag{15.26}$$

The nth functional derivative of the functional (15.21) is

$$\delta^n G_N[f][h] = \frac{N!}{(N-n)!} \int f^{N-n}(x) h^n(x) dx. \tag{15.27}$$

15.4 Functional Taylor Series

It follows from the Taylor-series theorem (section 4.6) that

$$e^{\delta} G[f][h] = \sum_{n=0}^{\infty} \frac{\delta^n}{n!} G[f][h] = \sum_{n=0}^{\infty} \frac{1}{n!}\, \frac{d^n}{d\epsilon^n} G[f + \epsilon h] \Bigg|_{\epsilon=0} = G[f + h] \tag{15.28}$$

which illustrates an advantage of the present mathematical notation.

The functional $S_0[q]$ of Eq.(15.1) provides a simple example of the func-