Let's now compute the functional derivative of the action (15.2), which involves the square of the time-derivative $\dot{q}(t)$ and the potential energy $V(q(t))$

$$\begin{aligned}
\delta S[q][h] &= \frac{d}{d\epsilon} S[q+\epsilon h]\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} \int dt \left[\frac{m}{2} \left(\dot{q}(t) + \epsilon \dot{h}(t) \right)^2 - V(q(t) + \epsilon h(t)) \right] \Big|_{\epsilon=0} \\
&= \int dt \left[m\dot{q}(t)\dot{h}(t) - V'(q(t))h(t) \right] \\
&= \int dt \left[-m\ddot{q}(t) - V'(q(t)) \right] h(t)
\end{aligned} \tag{15.11}$$

where we once again have integrated by parts and used suitable boundary conditions to drop the surface terms. In physics notation, this is

$$\frac{\delta S[q]}{\delta q(t)} = \int dt' \left[-m\ddot{q}(t') - V'(q(t')) \right] \delta(t'-t) = -m\ddot{q}(t) - V'(q(t)). \tag{15.12}$$

In these terms, the stationarity of the action $S[q]$ is the vanishing of its functional derivative either in the form

$$\delta S[q][h] = 0 \tag{15.13}$$

for arbitrary functions $h(t)$ (that vanish at the end points of the interval) or equivalently in the form

$$\frac{\delta S[q]}{\delta q(t)} = 0 \tag{15.14}$$

which is Lagrange's equation of motion

$$m\ddot{q}(t) = -V'(q(t)). \tag{15.15}$$

Physicists also use the compact notation

$$\frac{\delta^2 Z[j]}{\delta j(y) \delta j(z)} \equiv \frac{\partial^2 Z[j + \epsilon \delta_y + \epsilon' \delta_z]}{\partial \epsilon \, \partial \epsilon'} \Big|_{\epsilon=\epsilon'=0} \tag{15.16}$$

in which $\delta_y(x) = \delta(x-y)$ and $\delta_z(x) = \delta(x-z)$.

Example 15.1 (Shortest Path is a Straight Line) On a plane, the length of the path $(x, y(x))$ from (x_0, y_0) to (x_1, y_1) is

$$L[y] = \int_{x_0}^{x_1} \sqrt{dx^2 + dy^2} = \int_{x_0}^{x_1} \sqrt{1 + y'^2} \, dx. \tag{15.17}$$

The shortest path $y(x)$ minimizes this length $L[y]$, so

$$\delta L[y][h] = \frac{d}{d\epsilon} L[y+\epsilon h]\bigg|_{\epsilon=0} = \frac{d}{d\epsilon} \int_{x_0}^{x_1} \sqrt{1+(y'+\epsilon h')^2}\, dx\bigg|_{\epsilon=0}$$
$$= \int_{x_0}^{x_1} \frac{y'h'}{\sqrt{1+y'^2}}\, dx = -\int_{x_0}^{x_1} h \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\, dx = 0 \tag{15.18}$$

since $h(x_0) = h(x_1) = 0$. This can vanish for arbitrary $h(x)$ only if

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0 \tag{15.19}$$

which implies $y'' = 0$. Thus $y(x)$ is a straight line, $y = mx + b$. □

15.3 Higher-Order Functional Derivatives

The second functional derivative is

$$\delta^2 G[f][h] = \frac{d^2}{d\epsilon^2}\, G[f+\epsilon h]|_{\epsilon=0}\,. \tag{15.20}$$

So if $G_N[f]$ is the functional

$$G_N[f] = \int f^N(x)dx \tag{15.21}$$

then

$$\delta^2 G_N[f][h] = \frac{d^2}{d\epsilon^2}\, G_N[f+\epsilon h]|_{\epsilon=0}$$
$$= \frac{d^2}{d\epsilon^2} \int (f(x)+\epsilon h(x))^N\, dx\bigg|_{\epsilon=0}$$
$$= \frac{d^2}{d\epsilon^2} \int \binom{N}{2} \epsilon^2 h^2(x) f^{N-2}(x)\, dx\bigg|_{\epsilon=0}$$
$$= N(N-1) \int f^{N-2}(x) h^2(x) dx. \tag{15.22}$$

Example 15.2 ($\delta^2 S_0$) The second functional derivative of the action $S_0[q]$ (15.1) is

$$\delta^2 S_0[q][h] = \frac{d^2}{d\epsilon^2} \int_{t_1}^{t_2} dt\, \frac{m}{2} \left(\frac{dq(t)}{dt} + \epsilon \frac{dh(t)}{dt}\right)^2\bigg|_{\epsilon=0}$$
$$= \int_{t_1}^{t_2} dt\, m \left(\frac{dh(t)}{dt}\right)^2 \geq 0 \tag{15.23}$$