of the characteristic functions

$$\tilde{P}_j(k/N) = \int e^{ikx_j/N}\, P_j(x_j)\, dx_j \tag{13.213}$$

of the probability distributions $P_1(x_1), \ldots, P_N(x_N)$.

The Taylor series (13.187) for each characteristic function is

$$\tilde{P}_j(k/N) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!\, N^n}\, \mu_{nj} \tag{13.214}$$

and so for big N we can use the approximation

$$\tilde{P}_j(k/N) \approx 1 + \frac{ik}{N}\, \mu_j - \frac{k^2}{2N^2}\, \mu_{2j} \tag{13.215}$$

in which $\mu_{2j} = \sigma_j^2 + \mu_j^2$ by the formula (13.22) for the variance. So we have

$$\tilde{P}_j(k/N) \approx 1 + \frac{ik}{N}\, \mu_j - \frac{k^2}{2N^2}\, (\sigma_j^2 + \mu_j^2) \tag{13.216}$$

or for large N

$$\tilde{P}_j(k/N) \approx \exp\left(\frac{ik}{N}\mu_j - \frac{k^2}{2N^2}\sigma_j^2\right). \tag{13.217}$$

Thus as $N \to \infty$, the characteristic function (13.212) for the variable y converges to

$$\begin{aligned}\tilde{P}^{(N)}(k) &= \prod_{j=1}^{N} \tilde{P}_j(k/N) \; = \prod_{j=1}^{N} \exp\left(\frac{ik}{N}\mu_j - \frac{k^2}{2N^2}\sigma_j^2\right) \\ &= \exp\left[\sum_{j=1}^{N}\left(\frac{ik}{N}\mu_j - \frac{k^2}{2N^2}\sigma_j^2\right)\right] = \exp\left(i\mu_y k - \frac{1}{2}\sigma_y^2 k^2\right)\end{aligned} \tag{13.218}$$

which is the characteristic function (13.176) of a gaussian (13.175) with mean and variance

$$\mu_y = \frac{1}{N}\sum_{j=1}^{N}\mu_j \quad \text{and} \quad \sigma_y^2 = \frac{1}{N^2}\sum_{j=1}^{N}\sigma_j^2. \tag{13.219}$$

The inverse Fourier transform (13.174) now gives the probability distribution $P^{(N)}(y)$ for the average $y = (x_1 + x_2 + \cdots + x_N)/N$ as

$$P^{(N)}(y) = \int_{-\infty}^{\infty} e^{-iky}\, \tilde{P}^{(N)}(k)\, \frac{dk}{2\pi} \tag{13.220}$$