Δq_i, one may show (exercise 12.10) that this sum of areas remains constant

$$\frac{d}{dt}\, d\omega^1(\delta p, \delta q; \Delta p, \Delta q) = 0 \tag{12.77}$$

along the trajectories in phase space (Gutzwiller, 1990, chap. 7). □

Example 12.12 (The Curl) We saw in example 12.7 that the 1-form (12.50) of a vector field $\boldsymbol{A}$ is $\omega_A = A_1\, h_1\, dx_1 + A_2\, h_2\, dx_2 + A_3\, h_3\, dx_3$ in which the h_k's are those that determine (12.44) the squared length $ds^2 = h_k^2\, dx_k^2$ of the triply orthogonal coordinate system with unit vectors $\hat{\boldsymbol{e}}_{\boldsymbol{1}}$, $\hat{\boldsymbol{e}}_{\boldsymbol{2}}$, $\hat{\boldsymbol{e}}_{\boldsymbol{3}}$. So the exterior derivative of the 1-form ω_A is

$$\begin{aligned} d\omega_A &= \sum_{i,k=1}^{3} \partial_k (A_i\, h_i)\, dx_k \wedge dx_i \\ &= \left[\frac{\partial (A_3\, h_3)}{\partial x_2} - \frac{\partial (A_2\, h_2)}{\partial x_3} \right] dx_2 \wedge dx_3 \\ &\quad + \left[\frac{\partial (A_2\, h_2)}{\partial x_1} - \frac{\partial (A_1\, h_1)}{\partial x_2} \right] dx_1 \wedge dx_2 \\ &\quad + \left[\frac{\partial A_1\, h_1}{\partial x_3} - \frac{\partial (A_3\, h_3)}{\partial x_1} \right] dx_3 \wedge dx_1 \equiv \omega_{\nabla \times \boldsymbol{A}}. \end{aligned} \tag{12.78}$$

Comparison with Eq. (12.52) shows that the curl of $\boldsymbol{A}$ is

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{A} &= \frac{1}{h_2\, h_3} \left(\frac{\partial A_3\, h_3}{\partial x_2} - \frac{\partial A_2\, h_2}{\partial_3} \right) dx_2 \wedge dx_3\, \hat{\boldsymbol{e}}_{\boldsymbol{1}} + \dots \\ &= \frac{1}{h_1\, h_2\, h_3} \begin{vmatrix} h_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} & h_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} & h_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}} \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \\ &= \frac{1}{h_1 h_2 h_3} \sum_{i,j,k=1}^{3} \epsilon_{ijk} h_i\, \hat{\boldsymbol{e}}_{\boldsymbol{i}}\, \frac{\partial (A_k h_k)}{\partial x_j} \end{aligned} \tag{12.79}$$

as we saw in (11.240). This formula gives our earlier expressions for the curl in cylindrical and spherical coordinates (11.241 & 11.242). □

Example 12.13 (The Divergence) We have seen in equations (12.48, 12.49, & 12.52) that the 2-form $\omega_A(U, V) = A \cdot (U \times V)$ of the vector field $\boldsymbol{A} = A_1 \hat{\boldsymbol{e}}_{\boldsymbol{1}} + A_2 \hat{\boldsymbol{e}}_{\boldsymbol{2}} + A_3 \hat{\boldsymbol{e}}_{\boldsymbol{3}}$ is

$$\omega_A^2 = A_1\, h_2\, h_3\; dx_2 \wedge dx_3 + A_2\, h_3\, h_1\; dx_3 \wedge dx_1 + A_3\, h_1\, h_2\; dx_1 \wedge dx_2. \tag{12.80}$$