There are two ways of thinking about differential forms. The Russian literature views a manifold as embedded in $\mathbb{R}^n$ and so is somewhat more straightforward. We will discuss it first.

The Russian Way: Suppose $x(t)$ is a curve with $x(0) = x$ on some **manifold** M, and $f(x(t))$ is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ that maps points $x(t)$ into numbers. Then the **differential** $df(\dot{x}(t))$ maps $\dot{x}(t)$ at x into

$$df\left(\frac{d}{dt}x(t)\right) \equiv \frac{d}{dt}f(x(t)) = \sum_{j=1}^{n} \dot{x}(t)_j \frac{\partial f(x(t))}{\partial x_j} = \dot{x}(t) \cdot \nabla f(x(t)) \quad (12.18)$$

all at $t = 0$. As physicists, we think of df as a number—the change in the function $f(x)$ when its argument x is changed by dx. Russian mathematicians think of df as a linear map of tangent vectors $\dot{x}$ at x into numbers. Since this map is linear, we may multiply the definition (12.18) by dt and arrive at the more familiar formula

$$dt\, df\left(\frac{d}{dt}x(t)\right) = df\left(dt\frac{d}{dt}x(t)\right) = df\,(dx(t)) = dx(t) \cdot \nabla f(x(t)) \quad (12.19)$$

all at $t = 0$. So

$$df(dx) = dx \cdot \nabla f. \quad (12.20)$$

is the physicist's df.

Since the differential df is a linear map of vectors $\dot{x}(0)$ into numbers, it is a 1-form; since it is defined on vectors like $\dot{x}(0)$, it is a **differential 1-form**. The term *differential 1-form* underscores the fact that the actual value of the differential df depends upon the vector $\dot{x}(0)$ and the point $x = x(0)$. Mathematicians call the space of vectors $\dot{x}(0)$ at the point $x = x(0)$ the **tangent space** TM_x. They say df is a smooth map of the **tangent bundle** TM, which is the union of the tangent spaces for all points x in the manifold M, to the real line, so $df : TM \to \mathbb{R}$.

In the special case in which $f(x) = x_i(x) = x_i$, the differential $dx_i(\dot{x}(t))$ by (12.18) is

$$dx_i(\dot{x}(t)) = \sum_{j=1}^{n} \dot{x}_j(t)\frac{\partial x_i(x)}{\partial x_j} = \sum_{j=1}^{n} \dot{x}_j(t)\frac{\partial x_i}{\partial x_j} = \sum_{j=1}^{n} \dot{x}(t)_j\, \delta_{ij} = \dot{x}_i(t). \quad (12.21)$$

These dx_i's are the **basic differentials**. Using A for the vector $\dot{x}(t)$, we find from our definition (12.18) that

$$dx_i(A) = \sum_{j=1}^{n} A_j \frac{\partial x_i}{\partial x_j} = \sum_{j=1}^{n} A_j \delta_{ij} = A_i \quad (12.22)$$