With calendars chosen so that $a(0) = 0$, this last equation (11.449) tells us that for a flat universe ($k = 0$)

$$a(t) = (2f\,t)^{1/2} \tag{11.450}$$

while for a closed universe ($k = 1$)

$$a(t) = \sqrt{f^2 - (t-f)^2} \tag{11.451}$$

and for an open universe ($k = -1$)

$$a(t) = \sqrt{(t+f)^2 - f^2} \tag{11.452}$$

as we saw in (6.422). The scale factor (11.451) of a closed universe of radiation has a maximum $a = f$ at $t = f$ and falls back to zero at $t = 2f$. □

Example 11.28 ($w = 0$, The Era of Matter) A universe composed only of **dust** or **non-relativistic collisionless matter** has no pressure. Thus $p = w\rho = 0$ with $\rho \neq 0$, and so $w = 0$. Conservation of energy (11.433), or equivalently (11.434), implies that the energy density falls with the volume

$$\rho = \overline{\rho} \left(\frac{\overline{a}}{a} \right)^3 . \tag{11.453}$$

As the scale factor $a(t)$ increases, the matter energy density, which falls as $1/a^3$, eventually dominates the radiation energy density, which falls as $1/a^4$. This happened in our universe about 50,000 years after inflation at a temperature of $T = 9,400$ K or $kT = 0.81$ eV. Were baryons most of the matter, the era of radiation dominance would have lasted for a few hundred thousand years. But the kind of matter that we know about, which interacts with photons, is only about 15% of the total; the rest—an unknown substance called **dark matter**—shortened the era of radiation dominance by nearly 2 million years.

Since $\rho \propto 1/a^3$, the quantity

$$m^2 = \frac{4\pi G \rho a^3}{3} \tag{11.454}$$

is a constant. For a matter-dominated universe, the Friedmann equations (11.413 & 11.414) then are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p) = -\frac{4\pi G\rho}{3} \quad \text{or} \quad \ddot{a} = -\frac{m^2}{a^2} \tag{11.455}$$