If we multiply by this equation of motion by g^{rk} and note that $g_{ik,\ell} u^i u^\ell = g_{\ell k,i} u^i u^\ell$, then we find

$$0 = \frac{du^r}{d\tau} + \tfrac{1}{2} g^{rk} \left(g_{ik,\ell} + g_{\ell k,i} - g_{i\ell,k} \right) u^i u^\ell. \tag{11.323}$$

So using the symmetry $g_{i\ell} = g_{\ell i}$ and the formula (11.255) for $\Gamma^r_{i\ell}$, we get

$$0 = \frac{du^r}{d\tau} + \Gamma^r_{i\ell}\, u^i u^\ell \qquad \text{or} \qquad 0 = \frac{d^2 x^r}{d\tau^2} + \Gamma^r_{i\ell} \frac{dx^i}{d\tau} \frac{dx^\ell}{d\tau} \tag{11.324}$$

which is the geodesic equation. In empty space, particles fall along geodesics **independently of their masses**.

The right-hand side of the geodesic equation (11.324) is a contravariant vector because (Weinberg, 1972) under general coordinate transformations, the inhomogeneous terms arising from $\ddot{x}^r$ cancel those from $\Gamma^r_{i\ell} \dot{x}^i \dot{x}^\ell$. Here and often in what follows we'll use dots to mean proper-time derivatives.

The action for a particle of mass m and charge q in a gravitational field $\Gamma^r_{i\ell}$ and an electromagnetic field A_i is

$$S = -m \int \left(- g_{i\ell} dx^i dx^\ell \right)^{\frac{1}{2}} + q \int_{\tau_1}^{\tau_2} A_i(x)\, dx^i \tag{11.325}$$

because the interaction $q \int A_i dx^i$ is invariant under general coordinate transformations. By (11.315 & 11.321), the first-order change in S is

$$\delta S = m \int_{\tau_1}^{\tau_2} \left[\tfrac{1}{2} g_{i\ell,k} u^i u^\ell - g_{ik,\ell} u^i u^\ell - g_{ik} \frac{du^i}{d\tau} + q \left(A_{i,k} - A_{k,i} \right) u^i \right] \delta x^k d\tau \tag{11.326}$$

and so by combining the Lorentz force law (11.316) and the geodesic equation (11.324) and by writing $F^{ri} \dot{x}_i$ as $F^r{}_i\, \dot{x}^i$, we have

$$0 = \frac{d^2 x^r}{d\tau^2} + \Gamma^r_{i\ell} \frac{dx^i}{d\tau} \frac{dx^\ell}{d\tau} - \frac{q}{m} F^r{}_i \frac{dx^i}{d\tau} \tag{11.327}$$

as the equation of motion of a particle of mass m and change q. It is striking how nearly perfect the electromagnetism of Faraday and Maxwell is.

The action of the electromagnetic field interacting with an electric current j^k in a gravitational field is

$$S = \int \left[-\tfrac{1}{4} F_{k\ell} F^{k\ell} + \mu_0 A_k j^k \right] \sqrt{g}\, d^4 x \tag{11.328}$$

in which $\sqrt{g}\, d^4 x$ is the invariant volume element. After an integration by parts, the first-order change in the action is

$$\delta S = \int \left[- \frac{\partial}{\partial x^\ell} \left(F^{k\ell} \sqrt{g} \right) + \mu_0 j^k \sqrt{g} \right] \delta A_k\, d^4 x, \tag{11.329}$$

and so the inhomogeneous Maxwell equations in a gravitational field are

$$\frac{\partial}{\partial x^\ell}\left(\sqrt{g}\, F^{k\ell}\right) = \mu_0 \sqrt{g}\, j^k. \tag{11.330}$$

11.39 The Principle of Equivalence

The **principle of equivalence** says that in any gravitational field, one may choose free-fall coordinates in which all physical laws take the same form as in special relativity without acceleration or gravitation—at least over a suitably small volume of space-time. Within this volume and in these coordinates, things behave as they would at rest deep in empty space far from any matter or energy. The volume must be small enough so that the gravitational field is constant throughout it.

Example 11.21 (Elevators) When a modern elevator starts going down from a high floor, it accelerates downward at something less than the local acceleration of gravity. One feels less pressure on one's feet; one feels lighter. (This is as close to free fall as I like to get.) After accelerating downward for a few seconds, the elevator assumes a constant downward speed, and then one feels the normal pressure of one's weight on one's feet. The elevator seems to be slowing down for a stop, but actually it has just stopped accelerating downward.

If in those first few seconds the elevator really were falling, then the physics in it would be the same as if it were at rest in empty space far from any gravitational field. A clock in it would tick as fast as it would at rest in the absence of gravity. □

The transformation from arbitrary coordinates x^k to free-fall coordinates y^i changes the metric $g_{j\ell}$ to the diagonal metric η_{ik} of flat space-time $\eta = \text{diag}(-1, 1, 1, 1)$, which has two indices and is not a Levi-Civita tensor. Algebraically, this transformation is a congruence (1.308)

$$\eta_{ik} = \frac{\partial x^j}{\partial y^i}\, g_{j\ell}\, \frac{\partial x^\ell}{\partial y^k}. \tag{11.331}$$

The geodesic equation (11.324) follows from the **principle of equivalence** (Weinberg, 1972; Hobson et al., 2006). Suppose a particle is moving under the influence of gravitation alone. Then one may choose free-fall coordinates $y(x)$ so that the particle obeys the force-free equation of motion