If the particle is moving under the influence of a potential $V(\boldsymbol{x})$, then the action is

$$S = \int_{t_1}^{t_2} \left(\frac{m}{2} \dot{\boldsymbol{x}}^2 - V(\boldsymbol{x}) \right) dt. \tag{11.303}$$

Since $\delta V(\boldsymbol{x}) = \nabla V(\boldsymbol{x}) \cdot \delta \boldsymbol{x}$, the principle of stationary action requires that

$$0 = \delta S = \int_{t_1}^{t_2} (-m\ddot{\boldsymbol{x}} - \nabla V) \cdot \delta \boldsymbol{x} \, dt \tag{11.304}$$

or

$$m\ddot{\boldsymbol{x}} = -\nabla V \tag{11.305}$$

which is the classical equation of motion for a particle of mass m in a potential V.

The action for a free particle of mass m in special relativity is

$$S = \, - \, m \int_{\tau_1}^{\tau_2} d\tau = \, - \int_{t_1}^{t_2} m\sqrt{1 - \dot{\boldsymbol{x}}^2} \, dt \tag{11.306}$$

where $c = 1$ and $\dot{\boldsymbol{x}} = d\boldsymbol{x}/dt$. The requirement of stationary action is

$$0 = \delta S = \, - \, \delta \int_{t_1}^{t_2} m\sqrt{1 - \dot{\boldsymbol{x}}^2} \, dt = m \int_{t_1}^{t_2} \frac{\dot{\boldsymbol{x}} \cdot \delta \dot{\boldsymbol{x}}}{\sqrt{1 - \dot{\boldsymbol{x}}^2}} \, dt \tag{11.307}$$

But $1/\sqrt{1 - \dot{\boldsymbol{x}}^2} = dt/d\tau$ and so

$$\begin{aligned} 0 = \delta S &= m \int_{t_1}^{t_2} \frac{d\boldsymbol{x}}{dt} \cdot \frac{d\delta \boldsymbol{x}}{dt} \frac{dt}{d\tau} \, dt = m \int_{\tau_1}^{\tau_2} \frac{d\boldsymbol{x}}{dt} \cdot \frac{d\delta \boldsymbol{x}}{dt} \frac{dt}{d\tau} \frac{dt}{d\tau} \, d\tau \\ &= m \int_{\tau_1}^{\tau_2} \frac{d\boldsymbol{x}}{d\tau} \cdot \frac{d\delta \boldsymbol{x}}{d\tau} \, d\tau. \end{aligned} \tag{11.308}$$

So integrating by parts, keeping in mind that $\delta \boldsymbol{x}(\tau_2) = \delta \boldsymbol{x}(\tau_1) = \mathbf{0}$, we have

$$0 = \delta S = m \int_{\tau_1}^{\tau_2} \left[\frac{d}{d\tau} \left(\frac{d\boldsymbol{x}}{d\tau} \cdot \delta \boldsymbol{x} \right) - \frac{d^2 \boldsymbol{x}}{d\tau^2} \cdot \delta \boldsymbol{x} \right] d\tau = -m \int_{\tau_1}^{\tau_2} \frac{d^2 \boldsymbol{x}}{d\tau^2} \cdot \delta \boldsymbol{x} \, d\tau. \tag{11.309}$$

To have this hold for arbitrary $\delta \boldsymbol{x}$, we need

$$\frac{d^2 \boldsymbol{x}}{d\tau^2} = \mathbf{0} \tag{11.310}$$

which is the equation of motion for a free particle in special relativity.