

11.27 Derivatives and Affine Connections

If $F(x)$ is a vector field, then its invariant description in terms of space-time-dependent basis vectors $e_i(x)$ is

$$F(x) = F^i(x) e_i(x). \quad (11.210)$$

Since the basis vectors $e_i(x)$ vary with x , the derivative of $F(x)$ contains two terms

$$\frac{\partial F}{\partial x^\ell} = \frac{\partial F^i}{\partial x^\ell} e_i + F^i \frac{\partial e_i}{\partial x^\ell}. \quad (11.211)$$

In general, the derivative of a vector e_i is not a linear combination of the basis vectors e_k . For instance, on the 2-dimensional surface of a sphere in 3-dimensions, the derivative

$$\frac{\partial e_\theta}{\partial \theta} = -\hat{r} \quad (11.212)$$

points to the sphere's center and isn't a linear combination of e_θ and e_ϕ .

The inner product of a derivative $\partial e_i / \partial x^\ell$ with a dual basis vector e^k is the **Levi-Civita affine connection**

$$\Gamma_{li}^k = e^k \cdot \frac{\partial e_i}{\partial x^\ell} \quad (11.213)$$

which relates spaces that are tangent to the manifold at infinitesimally separated points. It is called an *affine* connection because the different tangent spaces lack a common origin.

In terms of the affine connection (11.213), the inner product of the derivative of (11.211) with e^k is

$$e^k \cdot \frac{\partial F}{\partial x^\ell} = e^k \cdot \frac{\partial F^i}{\partial x^\ell} e_i + F^i e^k \cdot \frac{\partial e_i}{\partial x^\ell} = \frac{\partial F^k}{\partial x^\ell} + \Gamma_{li}^k F^i \quad (11.214)$$

a combination that is called a **covariant derivative** (section 11.30)

$$D_\ell F^k \equiv \nabla_\ell F^k \equiv \frac{\partial F^k}{\partial x^\ell} + \Gamma_{li}^k F^i = e^k \cdot \frac{\partial F}{\partial x^\ell}. \quad (11.215)$$

It is a second-rank mixed tensor.

Some physicists write the affine connection $\Gamma_{i\ell}^k$ as

$$\left\{ \begin{matrix} k \\ i\ell \end{matrix} \right\} = \Gamma_{i\ell}^k \quad (11.216)$$

and call it a **Christoffel symbol of the second kind**.