Since the vectors $e_i$ are orthogonal, the metric is diagonal

$$g_{ij} = e_i \cdot e_j = h_i^2 \delta_{ij}. \tag{11.154}$$

The inverse metric

$$g^{ij} = h_i^{-2} \delta_{ij} \tag{11.155}$$

raises indices. For instance, the dual vectors

$$e^{\,i} = g^{ij}\, e_j = h_i^{-2}\, e_i \quad \text{satisfy} \quad e^{\,i} \cdot e_k = \delta^i_k. \tag{11.156}$$

The invariant squared distance $dp^2$ between nearby points (11.143) is

$$dp^2 = dp \cdot dp = g_{ij}\, dx^i\, dx^j = h_i^2\, (dx^i)^2 \tag{11.157}$$

and the invariant volume element is

$$dV = d^n p = h_1 \dots h_n\, dx^1 \wedge \dots \wedge dx^n = g\, dx^1 \wedge \dots \wedge dx^n = g\, d^n x \tag{11.158}$$

in which $g = \sqrt{\det g_{ij}}$ is the square-root of the positive determinant of $g_{ij}$.

The important special case in which all the scale factors $h_i$ are unity is cartesian coordinates in euclidean space (section 11.5).

We also can use basis vectors $\hat{e}_i$ that are **orthonormal**. By (11.154 & 11.156), these vectors

$$\hat{e}_i = e_i/h_i = h_i\, e^{\,i} \quad \text{satisfy} \quad \hat{e}_i \cdot \hat{e}_j = \delta_{ij}. \tag{11.159}$$

In terms of them, a physical and invariant vector $V$ takes the form

$$V = e_i\, V^i = h_i\, \hat{e}_i\, V^i = e^{\,i}\, V_i = h_i^{-1}\, \hat{e}_i\, V_i = \hat{e}_i\, \overline{V}_i \tag{11.160}$$

where

$$\overline{V}_i \equiv h_i\, V^i = h_i^{-1}\, V_i \quad \text{(no sum)}. \tag{11.161}$$

The dot-product is then

$$V \cdot U = g_{ij}\, V^i\, U^j = \overline{V}_i\, \overline{U}_i. \tag{11.162}$$

In euclidian $n$-space, we even can choose coordinates $x^i$ so that the vectors $e_i$ defined by $dp = e_i\, dx^i$ are orthonormal. The metric tensor is then the $n \times n$ identity matrix $g_{ik} = e_i \cdot e_k = I_{ik} = \delta_{ik}$. But since this is euclidian $n$-space, we also can expand the $n$ fixed orthonormal cartesian unit vectors $\hat{\ell}$ in terms of the $e_i(x)$ which vary with the coordinates as $\hat{\ell} = e_i(x)(e_i(x) \cdot \hat{\ell})$.