Suppose $T(y)$ is a translation that takes a 4-vector x to $x+y$ and $T(z)$ is a translation that takes a 4-vector x to $x+z$. Then $T(z)T(y)$ and $T(y)T(z)$ both take x to $x+y+z$. So if a translation $T(y) = T(t, \boldsymbol{y})$ is represented by a unitary operator $U(t, \boldsymbol{y}) = \exp(iHt - i\boldsymbol{P} \cdot \boldsymbol{y})$, then the hamiltonian H and the momentum operator $\boldsymbol{P}$ commute with each other

$$[H, P^j] = 0 \quad \text{and} \quad [P^i, P^j] = 0. \tag{10.299}$$

We can figure out the commutation relations of H and $\boldsymbol{P}$ with the angular-momentum $\boldsymbol{J}$ and boost $\boldsymbol{K}$ operators by realizing that $P^a = (H, \boldsymbol{P})$ is a 4-vector. Let

$$U(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.300}$$

be the (infinite-dimensional) unitary operator that represents (in Hilbert space) the infinitesimal Lorentz transformation

$$L = I + \boldsymbol{\theta} \cdot \boldsymbol{R} + \boldsymbol{\lambda} \cdot \boldsymbol{B} \tag{10.301}$$

where $\boldsymbol{R}$ and $\boldsymbol{B}$ are the six 4×4 matrices (10.231 & 10.232). Then because P is a 4-vector under Lorentz transformations, we have

$$U^{-1}(\boldsymbol{\theta}, \boldsymbol{\lambda}) P U(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{+i\boldsymbol{\theta}\cdot\boldsymbol{J} + i\boldsymbol{\lambda}\cdot\boldsymbol{K}} P e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} = (I + \boldsymbol{\theta} \cdot \boldsymbol{R} + \boldsymbol{\lambda} \cdot \boldsymbol{B})\, P \tag{10.302}$$

or using (10.272)

$$\begin{aligned} (I + i\boldsymbol{\theta} \cdot \boldsymbol{J} + i\boldsymbol{\lambda} \cdot \boldsymbol{K})\, H\, (I - i\boldsymbol{\theta} \cdot \boldsymbol{J} - i\boldsymbol{\lambda} \cdot \boldsymbol{K}) &= H + \boldsymbol{\lambda} \cdot \boldsymbol{P} \\ (I + i\boldsymbol{\theta} \cdot \boldsymbol{J} + i\boldsymbol{\lambda} \cdot \boldsymbol{K})\, \boldsymbol{P}\, (I - i\boldsymbol{\theta} \cdot \boldsymbol{J} - i\boldsymbol{\lambda} \cdot \boldsymbol{K}) &= \boldsymbol{P} + H\boldsymbol{\lambda} + \boldsymbol{\theta} \wedge \boldsymbol{P}. \end{aligned} \tag{10.303}$$

Thus, one finds (exercise 10.42) that H is invariant under rotations, while $\boldsymbol{P}$ transforms as a 3-vector

$$[J_i, H] = 0 \quad \text{and} \quad [J_i, P_j] = i\epsilon_{ijk} P_k \tag{10.304}$$

and that

$$[K_i, H] = -iP_i \quad \text{and} \quad [K_i, P_j] = -i\delta_{ij} H. \tag{10.305}$$

By combining these equations with (10.285), one may write (exercise 10.44) the Lie algebra of the Poincaré group as

$$\begin{aligned} i[J^{ab}, J^{cd}] &= \eta^{bc} J^{ad} - \eta^{ac} J^{bd} - \eta^{da} J^{cb} + \eta^{db} J^{ca} \\ i[P^a, J^{bc}] &= \eta^{ab} P^c - \eta^{ac} P^b \\ [P^a, P^b] &= 0. \end{aligned} \tag{10.306}$$

Further Reading

The classic *Lie Algebras in Particle Physics* (Georgi, 1999), which inspired much of this chapter, is outstanding.

Exercises

10.1 Show that all $n \times n$ (real) orthogonal matrices O leave invariant the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$, that is, that if $x' = Ox$, then $x'^2 = x^2$.

10.2 Show that the set of all $n \times n$ orthogonal matrices forms a group.

10.3 Show that all $n \times n$ unitary matrices U leave invariant the quadratic form $|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$, that is, that if $x' = Ux$, then $|x'|^2 = |x|^2$.

10.4 Show that the set of all $n \times n$ unitary matrices forms a group.

10.5 Show that the set of all $n \times n$ unitary matrices with unit determinant forms a group.

10.6 Show that the matrix $D^{(j)}_{m'm}(g) = \langle j, m'|U(g)|j, m\rangle$ is unitary because the rotation operator $U(g)$ is unitary $\langle j, m'|U^\dagger(g)U(g)|j, m\rangle = \delta_{m'm}$.

10.7 Invent a group of order 3 and compute its multiplication table. For extra credit, prove that the group is unique.

10.8 Show that the relation (10.20) between two equivalent representations is an isomorphism.

10.9 Suppose that D_1 and D_2 are equivalent, finite-dimensional, irreducible representations of a group G so that $D_2(g) = SD_1(g)S^{-1}$ for all $g \in G$. What can you say about a matrix A that satisfies $D_2(g)\, A = A\, D_1(g)$ for all $g \in G$?

10.10 Find all components of the matrix $\exp(i\alpha A)$ in which

$$A = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \tag{10.307}$$

10.11 If $[A, B] = B$, find $e^{i\alpha A} B e^{-i\alpha A}$. Hint: what are the α-derivatives of this expression?

10.12 Show that the tensor-product matrix (10.31) of two representations D_1 and D_2 is a representation.

10.13 Find a 4×4 matrix S that relates the tensor-product representation $D_{\frac{1}{2} \otimes \frac{1}{2}}$ to the direct sum $D_1 \oplus D_0$.

10.14 Find the generators in the adjoint representation of the group with structure constants $f_{abc} = \epsilon_{abc}$ where a, b, c run from 1 to 3. *Hint:* The answer is three 3×3 matrices t_a, often written as L_a.

10.15 Show that the generators (10.90) satisfy the commutation relations (10.93).

10.16 Show that the demonstrated equation (10.98) implies the commutation relation (10.99).

10.17 Use the Cayley-Hamilton theorem (1.264) to show that the 3×3 matrix (10.96) that represents a right-handed rotation of θ radians about the axis $\boldsymbol{\theta}$ is given by (10.97).

10.18 Verify the mixed Jacobi identity (10.142).

10.19 For the group $SU(3)$, find the structure constants f_{123} and f_{231}.

10.20 Show that every 2×2 unitary matrix of unit determinant is a quaternion of unit norm.

10.21 Show that the quaternions as defined by (10.175) are closed under addition and multiplication and that the product xq is a quaternion if x is real and q is a quaternion.

10.22 Show that the one-sided derivative $f'(q)$ (10.184) of the quaternionic function $f(q) = q^2$ depends upon the direction along which $q' \to 0$.

10.23 Show that the generators (10.188) of $Sp(2n)$ obey commutation relations of the form (10.189) for some real structure constants f_{abc} and a suitably extended set of matrices A, A', ... and S_k, S'_k, ...

10.24 Show that for $0 < \epsilon \ll 1$, the real $2n \times 2n$ matrix $T = \exp(\epsilon JS)$ in which S is symmetric satisfies $T^{\mathsf{T}} JT = J$ (at least up to terms of order ϵ^2) and so is in $Sp(2n, R)$.

10.25 Show that the matrix T of (10.197) is in $Sp(2, R)$.

10.26 Use the parametrization (10.217) of the group $SU(2)$, show that the parameters $\boldsymbol{a}(\boldsymbol{c}, \boldsymbol{b})$ that describe the product $g(\boldsymbol{a}(\boldsymbol{c}, \boldsymbol{b})) = g(\boldsymbol{c})\, g(\boldsymbol{b})$ are those of (10.219).

10.27 Use formulas (10.219) and (10.212) to show that the left-invariant measure for $SU(2)$ is given by (10.220).

10.28 In tensor notation, which is explained in chapter 11, the condition (10.229) that $I + \omega$ be an infinitesimal Lorentz transformation reads $\left(\omega^{\mathsf{T}}\right)_b{}^a = \omega^a{}_b = -\,\eta_{bc}\,\omega^c{}_d\,\eta^{da}$ in which sums over c and d from 0 to 3 are understood. In this notation, the matrix η_{ef} lowers indices and η^{gh} raises them, so that $\omega_b{}^a = -\,\omega_{bd}\,\eta^{da}$. (Both η_{ef} and η^{gh} are numerically equal to the matrix η displayed in equation (10.222).) Multiply both sides of the condition (10.229) by $\eta_{ae} = \eta_{ea}$ and use the relation $\eta^{da}\,\eta_{ae} = \eta^d{}_e \equiv \delta^d{}_e$ to show that the matrix ω_{ab} with both indices lowered (or raised) is antisymmetric, that is,

$$\omega_{ba} = -\,\omega_{ab} \quad \text{and} \quad \omega^{ba} = -\,\omega^{ab}. \tag{10.308}$$

10.29 Show that the six matrices (10.231) and (10.232) satisfy the $SO(3,1)$ condition (10.229).

10.30 Show that the six generators $\boldsymbol{J}$ and $\boldsymbol{K}$ obey the commutations relations (10.234–10.236).

10.31 Show that if $\boldsymbol{J}$ and $\boldsymbol{K}$ satisfy the commutation relations (10.234–10.236) of the Lie algebra of the Lorentz group, then so do $\boldsymbol{J}$ and $-\boldsymbol{K}$.

10.32 Show that if the six generators $\boldsymbol{J}$ and $\boldsymbol{K}$ obey the commutation relations (10.234–10.236), then the six generators $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ obey the commutation relations (10.243).

10.33 Relate the parameter α in the definition (10.253) of the standard boost $B(p)$ to the 4-vector p and the mass m.

10.34 Derive the formulas for $D^{(1/2,0)}(\mathbf{0}, \alpha\,\hat{\boldsymbol{p}})$ given in equation (10.254).

10.35 Derive the formulas for $D^{(0,1/2)}(\mathbf{0}, \alpha\,\hat{\boldsymbol{p}})$ given in equation (10.271).

10.36 For infinitesimal complex $\boldsymbol{z}$, derive the 4-vector properties (10.255 & 10.272) of $(-I, \boldsymbol{\sigma})$ under $D^{(1/2,0)}$ and of $(I, \boldsymbol{\sigma})$ under $D^{(0,1/2)}$.

10.37 Show that under the unitary Lorentz transformation (10.257), the action density (10.258) is Lorentz covariant (10.259).

10.38 Show that under the unitary Lorentz transformation (10.273), the action density (10.274) is Lorentz covariant (10.275).

10.39 Show that under the unitary Lorentz transformations (10.257 & 10.273), the Majorana mass terms (10.266 & 10.279) are Lorentz covariant.

10.40 Show that the definitions of the gamma matrices (10.281) and of the generators (10.283) imply that the gamma matrices transform as a 4-vector under Lorentz transformations (10.284).

10.41 Show that (10.283) and (10.284) imply that the generators J^{ab} satisfy the commutation relations (10.285) of the Lorentz group.

10.42 Show that the spinor $\zeta = \sigma_2 \xi^*$ defined by (10.295) is right handed (10.273) if ξ is left handed (10.257).

10.43 Use (10.303) to get (10.304 & 10.305).

10.44 Derive (10.306) from (10.285, 10.299, 10.304, & 10.305).