The trace formula (10.65) gives us the ***SU*(3) structure constants** as

$$f_{abc} = -2i\mathrm{Tr}\left([t_a, t_b]t_c\right). \tag{10.166}$$

They are real and totally antisymmetric with $f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$, and $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$.

While no two generators of $SU(2)$ commute, two generators of $SU(3)$ do. In the representation (10.162,10.163), t_3 and t_8 are diagonal and so commute

$$[t_3, t_8] = 0. \tag{10.167}$$

They generate the **Cartan subalgebra** (section 10.26) of $SU(3)$.

10.25 *SU(3)* and quarks

The generators defined by Eqs.(10.163 & 10.162) give us the 3×3 representation

$$D(\alpha) = \exp\left(i\alpha_a t_a\right) \tag{10.168}$$

in which the sum $a = 1, 2, \ldots 8$ is over the eight generators t_a. This representation acts on complex 3-vectors and is called the **3**.

Note that if

$$D(\alpha_1)D(\alpha_2) = D(\alpha_3) \tag{10.169}$$

then the complex conjugates of these matrices obey the same multiplication rule

$$D^*(\alpha_1)D^*(\alpha_2) = D^*(\alpha_3) \tag{10.170}$$

and so form another representation of $SU(3)$. It turns out that (unlike in $SU(2)$) this representation is inequivalent to the **3**; it is the $\overline{\mathbf{3}}$.

There are three quarks with masses less than about 100 MeV/c^2—the u, d, and s quarks. The other three quarks c, b, and t are more massive by factors of 12, 45, and 173. Nobody knows why. Gell-Mann and Zweig suggested that the low-energy strong interactions were approximately invariant under unitary transformations of the three light quarks, which they represented by a **3**, and of the three light antiquarks, which they represented by a $\overline{\mathbf{3}}$. They imagined that the eight light pseudo-scalar mesons, that is, the three pions π^-, π^0, π^+, the neutral η, and the four kaons K^0, K^+, K^- $\overline{K}^0$, were composed of a quark and an antiquark. So they should transform as the tensor product

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{10.171}$$

They put the eight pseudo-scalar mesons into an **8**.

They imagined that the eight light baryons — the two nucleons N and P, the three sigmas Σ^-, Σ^0, Σ^+, the neutral lambda Λ, and the two cascades Ξ^- and Ξ^0 were each made of three quarks. They should transform as the tensor product

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{10.172}$$

They put the eight light baryons into one of these **8**'s. When they were writing these papers, there were nine spin-3/2 resonances with masses somewhat heavier than 1200 MeV/c^2 — four Δ's, three Σ^*'s, and two Ξ^*'s. They put these into the **10** and predicted the tenth and its mass. In 1964, a tenth spin-3/2 resonance, the Ω^-, was found with a mass close to their prediction of 1680 MeV/c^2, and by 1973 an MIT-SLAC team had discovered quarks inside protons and neutrons. (George Zweig, 1937–)

10.26 Cartan Subalgebra

In any Lie group, the maximum set of mutually commuting generators H_a generate the **Cartan subalgebra**

$$[H_a, H_b] = 0 \tag{10.173}$$

which is an abelian subalgebra. The number of generators in the Cartan subalgebra is the **rank** of the Lie algebra. The Cartan generators H_a can be simultaneously diagonalized, and their eigenvalues or diagonal elements are the **weights**

$$H_a|\mu, x, D\rangle = \mu_a|\mu, x, D\rangle \tag{10.174}$$

in which D labels the representation and x whatever other variables are needed to specify the state. The vector μ is the **weight vector**. The **roots** are the weights of the adjoint representation.

10.27 Quaternions

If z and w are any two complex numbers, then the 2×2 matrix

$$q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \tag{10.175}$$

is a quaternion. The quaternions are closed under addition and multiplication and under multiplication by a real number (exercise 10.21), but not