The boundary condition $u_\ell(kr_0) = 0$ fixes the ratio $v_\ell = d_\ell/c_\ell$ of the constants c_ℓ and d_ℓ. Thus for $\ell = 0$, Rayleigh's formulas (9.68 & 9.104) and the boundary condition say that $kr_0\, u_0(r_0) = c_0 \sin(kr_0) - d_0 \cos(kr_0) = 0$ or $d_0/c_0 = \tan kr_0$. The s-wave then is $u_0(kr) = c_0 \sin(kr - kr_0)/(kr \cos kr_0)$, which tells us that the tangent of the phase shift is $\tan \delta_0(k) = -kr_0$. By (9.90), the cross-section at low energy is $\sigma \approx 4\pi r_0^2$ or four times the classical value.

Similarly, one finds (exercise 9.28) that the tangent of the p-wave phase shift is

$$\tan \delta_1(k) = \frac{kr_0 \cos kr_0 - \sin kr_0}{\cos kr_0 - kr_0 \sin kr_0}. \tag{9.108}$$

For $kr_0 \ll 1$, we have $\delta_1(k) \approx -(kr_0)^3/3$; more generally, the ℓth phase shift is $\delta_\ell(k) \approx -(kr_0)^{2\ell+1}/\left\{(2\ell+1)[(2\ell-1)!!]^2\right\}$ for a potential of range r_0 at low energy $k \ll 1/r_0$. □

Further Reading

A great deal is known about Bessel functions. Students may find *Mathematical Methods for Physics and Engineering* (Riley et al., 2006) as well as the classics *A Treatise on the Theory of Bessel Functions* (Watson, 1995), *A Course of Modern Analysis* (Whittaker and Watson, 1927, chap. XVII), and *Methods of Mathematical Physics* (Courant and Hilbert, 1955) of special interest.

Exercises

9.1 Show that the series (9.1) for $J_n(\rho)$ satisfies Bessel's equation (9.4).

9.2 Show that the generating function $\exp(z(u - 1/u)/2)$ for Bessel functions is invariant under the substitution $u \to -1/u$.

9.3 Use the invariance of $\exp(z(u - 1/u)/2)$ under $u \to -1/u$ to show that $J_{-n}(z) = (-1)^n J_n(z)$.

9.4 By writing the generating function (9.5) as the product of the exponentials $\exp(zu/2)$ and $\exp(-z/2u)$, derive the expansion

$$\exp\left[\frac{z}{2}\left(u - u^{-1}\right)\right] = \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} \left(\frac{z}{2}\right)^{m+n} \frac{u^{m+n}}{(m+n)!} \left(-\frac{z}{2}\right)^{n} \frac{u^{-n}}{n!}. \tag{9.109}$$

9.5 From this expansion (9.109) of the generating function (9.5), derive the power-series expansion (9.1) for $J_n(z)$.

9.6 In the formula (9.5) for the generating function $\exp(z(u - 1/u)/2)$, replace u by $\exp i\theta$ and then derive the integral representation (9.6) for $J_n(z)$. Start with the interval $[-\pi, \pi]$.

9.7 From the general integral representation (9.6) for $J_n(z)$, derive the two integral formulas (9.7) for $J_0(z)$.

9.8 Show that the integral representations (9.6 & 9.7) imply that for any integer $n \neq 0$, $J_n(0) = 0$, while $J_0(0) = 1$.

9.9 By differentiating the generating function (9.5) with respect to u and identifying the coefficients of powers of u, derive the recursion relation

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z). \tag{9.110}$$

9.10 By differentiating the generating function (9.5) with respect to z and identifying the coefficients of powers of u, derive the recursion relation

$$J_{n-1}(z) - J_{n+1}(z) = 2\, J_n'(z). \tag{9.111}$$

9.11 Change variables to $z = ax$ and turn Bessel's equation (9.4) into the self-adjoint form (9.11).

9.12 If $y = J_n(ax)$, then equation (9.11) is $(xy')' + (xa^2 - n^2/x)y = 0$. Multiply this equation by xy', integrate from 0 to b, and so show that if $ab = z_{n,m}$ and $J_n(z_{n,m}) = 0$, then

$$2\int_0^b x\, J_n^2(ax)\, dx = b^2 J_n'^2(z_{n,m}) \tag{9.112}$$

which is the normalization condition (9.14).

9.13 Show that with $\lambda \equiv z^2/r_d^2$, the change of variables $\rho = zr/r_d$ and $u(r) = J_n(\rho)$ turns $-(r\,u')' + n^2\,u/r = \lambda\,r\,u$ into (9.25).

9.14 Use the formula (6.42) for the curl in cylindrical coordinates and the vacuum forms $\boldsymbol{\nabla}\times\boldsymbol{E} = -\dot{\boldsymbol{B}}$ and $\boldsymbol{\nabla}\times\boldsymbol{B} = \dot{\boldsymbol{E}}/c^2$ of the laws of Faraday and Maxwell-Ampère to derive the field equations (9.55).

9.15 Derive equations (9.56) from (9.55).

9.16 Show that $J_n(\sqrt{\omega^2/c^2 - k^2}\,\rho)e^{in\phi}e^{i(kz-\omega t)}$ is a traveling-wave solution (9.52) of the wave equations (9.57).

9.17 Find expressions for the nonzero TM fields in terms of the formula (9.59) for E_z.

9.18 Show that the TE field $E_z = 0$ and $B_z = J_n(\sqrt{\omega^2/c^2 - k^2}\,\rho)e^{in\phi}e^{i(kz-\omega t)}$ will satisfy the boundary conditions (9.54) if $\sqrt{\omega^2/c^2 - k^2}\,r$ is a zero $z'_{n,m}$ of J'_n.

9.19 Show that if ℓ is an integer and if $\sqrt{\omega^2/c^2 - \pi^2\ell^2/h^2}\, r$ is a zero $z'_{n,m}$ of J'_n, then the fields $E_z = 0$ and $B_z = J_n(z'_{n,m}\rho/r)\, e^{in\phi}\, \sin(\ell\pi z/h)\, e^{-i\omega t}$ satisfy both the boundary conditions (9.54) at $\rho = r$ and those (9.60) at $z = 0$ and h as well as the wave equations (9.57). Hint: Use Maxwell's equations $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$ and $\boldsymbol{\nabla} \times \boldsymbol{B} = \dot{\boldsymbol{E}}/c^2$ as in (9.55).

9.20 Show that the resonant frequencies of the TM modes of the cavity of example 9.4 are $\omega_{n,m,\ell} = c\sqrt{z^2_{n,m}/r^2 + \pi^2\ell^2/h^2}$.

9.21 By setting $n = \ell + 1/2$ and $j_\ell = \sqrt{\pi/2x}\, J_{\ell+1/2}$, show that Bessel's equation (9.4) implies that the spherical Bessel function j_ℓ satisfies (9.63).

9.22 Show that Rayleigh's formula (9.68) implies the recursion relation (9.69).

9.23 Use the recursion relation (9.69) to show by induction that the spherical Bessel functions $j_\ell(x)$ as given by Rayleigh's formula (9.68) satisfy their differential equation (9.66) which with $x = kr$ is

$$-x^2 j''_\ell - 2x j'_\ell + \ell(\ell+1) j_\ell = x^2 j_\ell. \tag{9.113}$$

Hint: start by showing that $j_0(x) = \sin(x)/x$ satisfies this equation. This problem involves some tedium.

9.24 Iterate the trick

$$\frac{d}{zdz}\int_{-1}^{1} e^{izx}\, dx = \frac{i}{z}\int_{-1}^{1} x e^{izx}\, dx = \frac{i}{2z}\int_{-1}^{1} e^{izx}\, d(x^2-1)$$
$$= -\frac{i}{2z}\int_{-1}^{1}(x^2-1)de^{izx} = \frac{1}{2}\int_{-1}^{1}(x^2-1)e^{izx}dx \tag{9.114}$$

to show that (Schwinger et al., 1998, p. 227)

$$\left(\frac{d}{zdz}\right)^{\ell}\int_{-1}^{1} e^{izx}\, dx = \int_{-1}^{1}\frac{(x^2-1)^\ell}{2^\ell \ell!}\, e^{izx}\, dx. \tag{9.115}$$

9.25 Use the expansions (9.76 & 9.77) to show that the inner product of the ket $|\boldsymbol{r}\rangle$ that represents a particle at $\boldsymbol{r}$ with polar angles θ and ϕ and the one $|\boldsymbol{k}\rangle$ that represents a particle with momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ with polar angles θ' and ϕ' is with $\boldsymbol{k}\cdot\boldsymbol{r} = kr\cos\theta$

$$\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{(2\pi)^{3/2}} e^{ikr\cos\theta} = \frac{1}{(2\pi)^{3/2}}\sum_{\ell=0}^{\infty}(2\ell+1)\, P_\ell(\cos\theta)\, i^\ell\, j_\ell(kr)$$
$$= \frac{1}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sqrt{\frac{2}{\pi}}\sum_{\ell=0}^{\infty} i^\ell\, j_\ell(kr)\, Y_{\ell,m}(\theta,\phi)\, Y^*_{\ell,m}(\theta',\phi'). \tag{9.116}$$