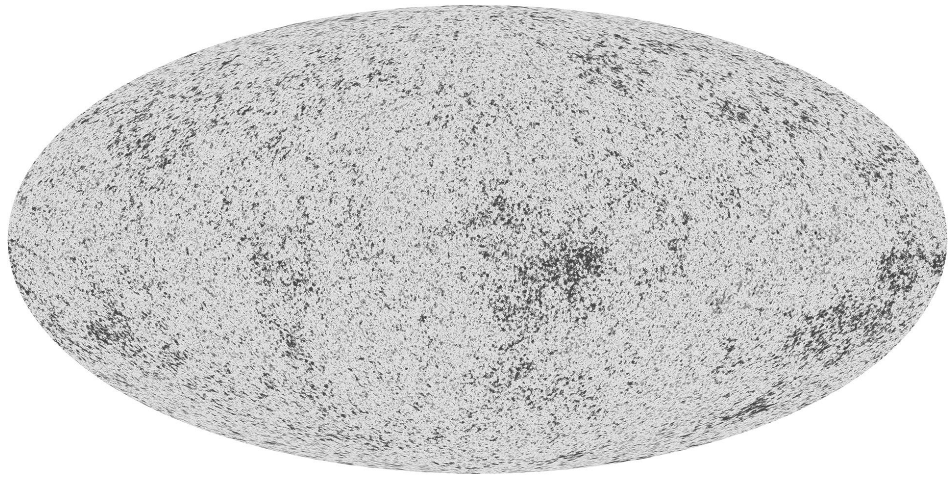

Figure 8.2 CMB temperature fluctuations over the celestial sphere as measured by the Planck satellite. The average temperature is 2.7255 K. White regions are warmer and black ones colder by about 0.0005 degrees. © ESA and the Planck Collaboration.

$P_\ell(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')$ of the cosine $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}'$ in which the polar angles of the unit vectors respectively are θ, ϕ and θ', ϕ' is the **addition theorem**

$$\begin{aligned} P_\ell(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}') &= \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta,\phi) Y^*_{\ell,m}(\theta',\phi') \\ &= \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell,m}(\theta,\phi) Y_{\ell,m}(\theta',\phi'). \end{aligned} \tag{8.121}$$

Example 8.8 (CMB Radiation) Instruments on the Wilkinson Microwave Anisotropy Probe (WMAP) and Planck satellites in orbit at the Lagrange point L_2 (in the Earth's shadow, 1.5×10^6 km farther from the Sun) have measured the temperature $T(\theta, \phi)$ of the cosmic microwave background (CMB) radiation as a function of the polar angles θ and ϕ in the sky as shown in Fig. 8.2. This radiation is photons last scattered when the visible universe became transparent at an age of 380,000 years and a temperature (3,000 K) cool enough for hydrogen atoms to be stable. This **initial tranparency** is usually (and inexplicably) called **recombination**.

Since the spherical harmonics $Y_{\ell,m}(\theta, \phi)$ are complete on the sphere, we

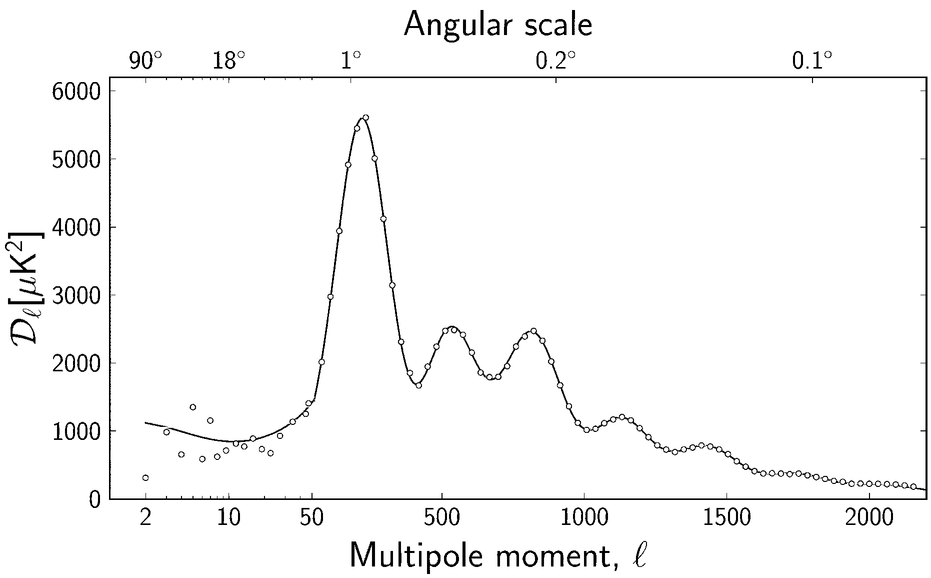

Figure 8.3 The power spectrum $\mathcal{D}_\ell = \ell(\ell+1)C_\ell/2\pi$ of the CMB temperature fluctuations in μK^2 as measured by the Planck Collaboration (arXiv:1303.5062) is plotted against the angular size and the multipole moment ℓ. The solid curve is the ΛCDM prediction.

can expand the temperature as

$$T(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m}\, Y_{\ell,m}(\theta, \phi) \tag{8.122}$$

in which the coefficients are by (8.117)

$$a_{\ell,m} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \;\; Y^*_{\ell,m}(\theta, \phi)\, T(\theta, \phi). \tag{8.123}$$

The average temperature $\overline{T}$ contributes only to $a_{0,0} = \overline{T} = 2.7255$ K. The other coefficients describe the difference $\Delta T(\theta, \phi) = T(\theta, \phi) - \overline{T}$. The angular power spectrum is

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell,m}|^2. \tag{8.124}$$

If we let the unit vector $\hat{\boldsymbol{n}}$ point in the direction θ, ϕ and use the addition theorem (8.121), then we can write the angular power spectrum as

$$C_\ell = \frac{1}{4\pi} \int d^2\hat{\boldsymbol{n}} \int d^2\hat{\boldsymbol{n}}' \; P_\ell(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')\, T(\hat{\boldsymbol{n}})\, T(\hat{\boldsymbol{n}}'). \tag{8.125}$$

In Fig. 8.3, the measured values (arXiv:1303.5062) of the power spectrum $\mathcal{D}_\ell = \ell(\ell+1)\,C_\ell/2\pi$ are plotted against ℓ for $1 < \ell < 1300$ with the angles and distances *decreasing* with ℓ. The power spectrum is a snapshot at the moment of initial transparency of the temperature distribution of the plasma of photons, electrons, and nuclei undergoing acoustic oscillations. In these oscillations, gravity opposes radiation pressure, and $|\Delta T(\theta, \phi)|$ is maximal both when the oscillations are most compressed and when they are most rarefied. Regions that gravity has squeezed to maximum compression at transparency form the first and highest peak. Regions that have bounced off their first maximal compression and that radiation pressure has expanded to minimum density at transparency form the second peak. Those at their second maximum compression at transparency form the third peak, and so forth.

The solid curve is the prediction of a model with **inflation, cold dark matter,** and a **cosmological constant** Λ. In this **model**, the age of the visible universe is 13.817 Gyr; the **Hubble constant** is $H_0 = 67.3$ km/sMpc; the energy density of the universe is enough to make the universe flat; and the fractions of the energy density due to baryons, **dark matter**, and **dark energy** are 4.9%, 26.6%, and 68.5% (Edwin Hubble 1889–1953). □

Much is known about Legendre functions. The books *A Course of Modern Analysis* (Whittaker and Watson, 1927, chap. XV) and *Methods of Mathematical Physics* (Courant and Hilbert, 1955) are outstanding.

Exercises

8.1 Use conditions (8.6) and (8.7) to find $P_0(x)$ and $P_1(x)$.

8.2 Using the Gram-Schmidt method (section 1.10) to turn the functions x^n into a set of functions $L_n(x)$ that are orthonormal on the interval $[-1, 1]$ with inner product (8.2), find $L_n(x)$ for $n = 0$, 1, 2, and 3. Isn't Rodrigues's formula (8.8) easier to use?

8.3 Derive the conditions (8.6–8.7) on the coefficients a_k of the Legendre polynomial $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

8.4 Use equations (8.6–8.7) to find $P_3(x)$ and $P_4(x)$.

8.5 In superscript notation (6.19), Leibniz's rule (4.46) for derivatives of products $u\,v$ of functions is

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)}\, v^{(k)}. \tag{8.126}$$