$T(z)$ are defined by its Laurent series

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \tag{5.336}$$

and the inverse relation

$$L_n = \frac{1}{2\pi i} \oint z^{n+1}\, T(z)\, dz. \tag{5.337}$$

Thus the commutator of two modes involves two loop integrals

$$[L_m, L_n] = \left[\frac{1}{2\pi i} \oint z^{m+1}\, T(z)\, dz, \frac{1}{2\pi i} \oint w^{n+1}\, T(w)\, dw \right] \tag{5.338}$$

which we may deform as long as we cross no poles. Let's hold w fixed and deform the z loop so as to keep the T's radially ordered when z is near w as in Fig. 5.10. The operator-product expansion of the radially ordered product $\mathcal{R}\{T(z)T(w)\}$ is

$$\mathcal{R}\{T(z)T(w)\} = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} T'(w) + \dots \tag{5.339}$$

in which the prime means derivative, c is a constant, and the dots denote terms that are analytic in z and w. The commutator introduces a minus sign that cancels most of the two contour integrals and converts what remains into an integral along a tiny circle C_w about the point w as in Fig. 5.10

$$[L_m, L_n] = \oint \frac{dw}{2\pi i} w^{n+1} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \right]. \tag{5.340}$$

After doing the z-integral, which is left as a homework exercise (5.43), one may use the Laurent series (5.336) for $T(w)$ to do the w-integral, which one may choose to be along a tiny circle about $w = 0$, and so find the commutator

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\, m(m^2-1)\, \delta_{m+n,0} \tag{5.341}$$

of the Virasoro algebra.

Exercises

5.1 Compute the two limits (5.6) and (5.7) of example 5.2 but for the function $f(x, y) = x^2 - y^2 + 2ixy$. Do the limits now agree? Explain.

5.2 Show that if $f(z)$ is analytic in a disk, then the integral of $f(z)$ around a tiny (isosceles) triangle of side $\epsilon \ll 1$ inside the disk is zero to order ϵ^2.