The second common convention puts the cut on the negative real axis. Here the value of θ is the same as in the first convention when the point z is in the upper half-plane. But in the lower half-plane, θ decreases from 0 to $-\pi$ as the point z moves clockwise from the positive real axis to just below the negative real axis, where $\theta = -\pi + \epsilon$. As one crosses the negative real axis moving clockwise or up, θ jumps by 2π while crossing the cut. The two conventions agree in the upper half-plane but differ by 2π in the lower half-plane.

Sometimes it is convenient to place the cut on the positive or negative imaginary axis — or along a line that makes an arbitrary angle with the real axis. In any particular calculation, we are at liberty to define the polar angle θ by placing the cut anywhere we like, but we must not change from one convention to another in the same computation.

5.16 Powers and Roots

The logarithm is the key to many other functions to which it passes its arbitrariness. For instance, any power a of $z = r\exp(i\theta)$ is defined as

$$z^a = \exp(a \ln z) = \exp\left[a\left(\ln r + i\theta + i2\pi n\right)\right] = r^a\, e^{ia\theta}\, e^{i2\pi n a}. \tag{5.183}$$

So z^a is not unique unless a is an integer. The square-root, for example,

$$\sqrt{z} = \exp\left[\tfrac{1}{2}(\ln r + i\theta + i2\pi n)\right] = \sqrt{r}\, e^{i\theta/2}\, e^{in\pi} = (-1)^n\,\sqrt{r}\, e^{i\theta/2} \tag{5.184}$$

changes sign when we change θ by 2π as we cross a cut. The m-th root

$$\sqrt[m]{z} = z^{1/m} = \exp\left(\frac{\ln z}{m}\right) \tag{5.185}$$

changes by $\exp(\pm 2\pi i/m)$ when we cross a cut and change θ by 2π. And when $a = u + iv$ is a complex number, z^a is

$$z^a = e^{a \ln z} = e^{(u+iv)(\ln r + i\theta + i2\pi n)} = r^{u+iv}\, e^{(-v+iu)(\theta+2\pi n)} \tag{5.186}$$

which changes by $\exp[2\pi(-v + iu)]$ as we cross a cut.

Example 5.27 (i^i) The number $i = \exp(i\pi/2 + i2\pi n)$ for any integer n. So the general value of i^i is $i^i = \exp[i(i\pi/2 + i2\pi n)] = \exp(-\pi/2 - 2\pi n)$. □

One can define a sequence of mth-root functions

$$\left(z^{1/m}\right)_n = \exp\left(\frac{\ln r + i(\theta + 2\pi n)}{m}\right) \tag{5.187}$$

one for each integer n. These functions are the **branches** of the mth-root