One may extend the definition (4.36) of n-factorial from positive integers to complex numbers by means of the integral formula

$$z! \equiv \int_0^\infty e^{-t}\, t^z\, dt \tag{4.53}$$

for $\mathrm{Re}\, z > -1$. In particular

$$0! = \int_0^\infty e^{-t}\, dt = 1 \tag{4.54}$$

which explains the definition (4.37). The factorial function $(z-1)!$ in turn defines the **gamma function** for $\mathrm{Re}\, z > 0$ as

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt = (z-1)! \tag{4.55}$$

as may be seen from (4.53). By differentiating this formula and integrating it by parts, we see that the gamma function satisfies the key identity

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty \left(-\frac{d}{dt}e^{-t}\right) t^z\, dt = \int_0^\infty e^{-t}\left(\frac{d}{dt}t^z\right) dt = \int_0^\infty e^{-t}\, z\, t^{z-1}\, dt \\ &= z\,\Gamma(z).\end{aligned} \tag{4.56}$$

Since $\Gamma(1) = 0! = 1$, we may use this identity (4.56) to extend the definition (5.102) of the gamma function in unit steps into the left half-plane

$$\Gamma(z) = \frac{1}{z}\,\Gamma(z+1) = \frac{1}{z}\,\frac{1}{z+1}\,\Gamma(z+2) = \frac{1}{z}\,\frac{1}{z+1}\,\frac{1}{z+2}\,\Gamma(z+3) = \dots \tag{4.57}$$

as long as we avoid the negative integers and zero. This extension leads to Euler's definition

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)}\, n^z \tag{4.58}$$

and to Weierstrass's (exercise 4.6)

$$\Gamma(z) = \frac{1}{z}\, e^{-\gamma z}\left[\prod_{n=1}^\infty \left(1+\frac{z}{n}\right) e^{-z/n}\right]^{-1} \tag{4.59}$$

(Karl Theodor Wilhelm Weierstrass, 1815–1897), and is an example of analytic continuation (section 5.12).

One may show (exercise 4.8) that another formula for $\Gamma(z)$ is

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt \tag{4.60}$$

for $\mathrm{Re}\, z > 0$ and that

$$\Gamma(n + \tfrac{1}{2}) = \frac{(2n)!}{n!\, 2^{2n}} \sqrt{\pi} \tag{4.61}$$

which implies (exercise 4.11) that

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi}. \tag{4.62}$$

Example 4.7 (Bessel Function of nonintegral index) We can use the gamma-function formula (4.55) for $n!$ to extend the definition (4.49) of the Bessel function of the first kind $J_n(\rho)$ to nonintegral values ν of the index n. Replacing n by ν and $(m+n)!$ by $\Gamma(m+\nu+1)$, we get

$$J_\nu(\rho) = \left(\frac{\rho}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, \Gamma(m+\nu+1)} \left(\frac{\rho}{2}\right)^{2m} \tag{4.63}$$

which makes sense even for complex values of ν. □

Example 4.8 (Spherical Bessel Function) The spherical Bessel function is defined as

$$j_\ell(\rho) \equiv \sqrt{\frac{\pi}{2\rho}}\, J_{\ell+1/2}(\rho). \tag{4.64}$$

For small values of its argument $|\rho| \ll 1$, the first term in the series (4.63) dominates and so (exercise 4.7)

$$j_\ell(\rho) \approx \frac{\sqrt{\pi}}{2} \left(\frac{\rho}{2}\right)^\ell \frac{1}{\Gamma(\ell+3/2)} = \frac{\ell\,!\,(2\rho)^\ell}{(2\ell+1)!} = \frac{\rho^\ell}{(2\ell+1)!!} \tag{4.65}$$

as one may show by repeatedly using the key identity $\Gamma(z+1) = z\,\Gamma(z)$. □

4.6 Taylor Series

If the function $f(x)$ is a real-valued function of a real variable x with a continuous Nth derivative, then Taylor's expansion for it is

$$\begin{aligned} f(x+a) &= f(x) + a f'(x) + \frac{a^2}{2} f''(x) + \cdots + \frac{a^{N-1}}{(N-1)!} f^{(N-1)} + E_N \\ &= \sum_{n=0}^{N-1} \frac{a^n}{n!} f^{(n)}(x) + E_N \end{aligned} \tag{4.66}$$