Interchanging and rearranging, we have

$$f(x) = \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} \right) f(y)\, dy. \tag{2.113}$$

But $f(x)$ and the phases e^{inx} are **periodic** with period 2π, so we also have

$$f(x + 2\pi\ell) = \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} \right) f(y)\, dy. \tag{2.114}$$

Thus we arrive at the **Dirac comb**

$$\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} = \sum_{\ell=-\infty}^{\infty} \delta(x - y - 2\pi\ell) \tag{2.115}$$

or more simply

$$\sum_{n=-\infty}^{\infty} \frac{e^{inx}}{2\pi} = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} \cos(nx)\right] = \sum_{\ell=-\infty}^{\infty} \delta(x - 2\pi\ell). \tag{2.116}$$

Example 2.13 (Dirac's Comb) The sum of the first 100,000 terms of this cosine series (2.116) for the Dirac comb is plotted for the interval $(-15, 15)$ in Fig. 2.11. Gibbs overshoots appear at the discontinuities. The integral of the first 100,000 terms from -15 to 15 is 5.0000. □

The stretched Dirac comb is

$$\sum_{n=-\infty}^{\infty} \frac{e^{2\pi in(x-y)/L}}{L} = \sum_{\ell=-\infty}^{\infty} \delta(x - y - \ell L). \tag{2.117}$$

Example 2.14 (Parseval's Identity) Using our formula (2.35) for the Fourier coefficients of a stretched interval, we can relate a sum of products $f_n^* \, g_n$ of the Fourier coefficients of the functions $f(x)$ and $g(x)$ to an integral of the product $f^*(x)\, g(x)$

$$\sum_{n=-\infty}^{\infty} f_n^* \, g_n = \sum_{n=-\infty}^{\infty} \int_0^L dx \, \frac{e^{i2\pi nx/L}}{\sqrt{L}} \, f^*(x) \int_0^L dy \, \frac{e^{-i2\pi ny/L}}{\sqrt{L}} \, g(y). \tag{2.118}$$

This sum contains Dirac's comb (2.117) and so

$$\begin{aligned} \sum_{n=-\infty}^{\infty} f_n^* \, g_n &= \int_0^L dx \int_0^L dy \, f^*(x)\, g(y) \, \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{i2\pi n(x-y)/L} \\ &= \int_0^L dx \int_0^L dy \, f^*(x)\, g(y) \sum_{\ell=-\infty}^{\infty} \delta(x - y - \ell L). \end{aligned} \tag{2.119}$$