that is nonnegative when the matrices are the same

$$(A, A) = \mathrm{Tr} A^\dagger A = \sum_{i=1}^{N} \sum_{j=1}^{L} A_{ij}^* A_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{L} |A_{ij}|^2 \geq 0 \tag{1.87}$$

which is zero only when $A = 0$. So this inner product is positive definite. □

A vector space with a positive-definite inner product (1.73–1.76) is called an **inner-product space**, a **metric space**, or a **pre-Hilbert space**.

A sequence of vectors f_n is a **Cauchy sequence** if for every $\epsilon > 0$ there is an integer $N(\epsilon)$ such that $\|f_n - f_m\| < \epsilon$ whenever both n and m exceed $N(\epsilon)$. A sequence of vectors f_n **converges** to a vector f if for every $\epsilon > 0$ there is an integer $N(\epsilon)$ such that $\|f - f_n\| < \epsilon$ whenever n exceeds $N(\epsilon)$. An inner-product space with a norm defined as in (1.80) is **complete** if each of its Cauchy sequences converges to a vector in that space. A **Hilbert space** is a complete inner-product space. Every finite-dimensional inner-product space is complete and so is a Hilbert space. But the term *Hilbert space* more often is used to describe infinite-dimensional complete inner-product spaces, such as the space of all square-integrable functions (David Hilbert, 1862–1943).

Example 1.17 (The Hilbert Space of Square-Integrable Functions) For the vector space of functions (1.55), a natural inner product is

$$(f, g) = \int_a^b dx\, f^*(x) g(x). \tag{1.88}$$

The squared norm $\| f \|$ of a function $f(x)$ is

$$\| f \|^2 = \int_a^b dx\, |f(x)|^2. \tag{1.89}$$

A function is **square integrable** if its norm is finite. The space of all square-integrable functions is an inner-product space; it also is complete and so is a Hilbert space. □

Example 1.18 (Minkowski Inner Product) The Minkowski or Lorentz inner product (p, x) of two 4-vectors $p = (E/c, p_1, p_2, p_3)$ and $x = (ct, x_1, x_2, x_3)$ is $\boldsymbol{p} \cdot \boldsymbol{x} - Et$. It is indefinite, nondegenerate (1.79), and invariant under Lorentz transformations, and often is written as $p \cdot x$ or as $p\, x$. If p is the 4-momentum of a freely moving physical particle of mass m, then

$$p \cdot p = \boldsymbol{p} \cdot \boldsymbol{p} - E^2/c^2 = -c^2 m^2 \leq 0. \tag{1.90}$$

The Minkowski inner product satisfies the rules (1.73, 1.74, and 1.79), but