Its derivatives, which we'll call $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$, are

$$\mathcal{P}^\tau_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}} \tag{19.10}$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial L}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (X')^2 X'_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}}. \tag{19.11}$$

In terms of them, the change in the action is

$$\delta S = \int_{\tau_i}^{\tau_f}\int_0^{\sigma_1} \left[\frac{\partial}{\partial \tau}\left(\delta X^\mu \mathcal{P}^\tau_\mu\right) + \frac{\partial}{\partial \sigma}\left(\delta X^\mu \mathcal{P}^\sigma_\mu\right) - \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right)\right] d\tau\, d\sigma. \tag{19.12}$$

The total τ-derivative integrates to a term involving the variation δX^μ which we require to vanish at the initial and final values of τ. So we drop that term and find that the net change in the action is

$$\delta S = \int_{\tau_i}^{\tau_f} \left[\delta X^\mu \mathcal{P}^\sigma_\mu\right]_0^{\sigma_1} d\tau - \int_{\tau_i}^{\tau_f}\int_0^{\sigma_1} \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right) d\tau\, d\sigma. \tag{19.13}$$

Thus the equations of motion for the string are

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0, \tag{19.14}$$

but the action is stationary only if

$$\int \delta X^\mu(\tau, \sigma_1)\mathcal{P}^\sigma_\mu(\tau, \sigma_1) - \delta X^\mu(\tau, 0)\mathcal{P}^\sigma_\mu(\tau, 0)\, d\tau = 0. \tag{19.15}$$

Closed strings automatically satisfy this condition. Open strings satisfy it if they obey for each end σ_* of the string and each spacetime dimension μ the boundary condition

$$\delta X^\mu(\tau, \sigma_*)\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0 \quad \text{(no sum over } \mu\text{)}. \tag{19.16}$$

For spatial indices, $\mu > 0$, one can impose either the **free-endpoint** boundary condition

$$\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0 \tag{19.17}$$

or the **Dirichlet** boundary condition

$$\delta X^\mu(\tau, \sigma_*) = 0. \tag{19.18}$$

But since $\delta X^0 \propto \delta\tau \neq 0$, only the free-endpoint condition (19.17) makes sense for the time component, $\mu = 0$.

19.3 Regge Trajectories

The quantity $\mathcal{P}^\tau_\mu(\tau,\sigma)$ defined as the derivative (19.10) turns out to be the momentum density of the string. The angular momentum M_{12} of a string rigidly rotating in the x, y plane is

$$M_{12}(\tau) = \int_0^{\sigma_1} X_1 \mathcal{P}^\tau_2(\tau,\sigma) - X_2 \mathcal{P}^\tau_1(\tau,\sigma)\, d\sigma. \tag{19.19}$$

In a parametrization of the string with $\tau = t$ and $d\sigma$ proportional to the energy density dE of the string, the x, y coordinates of the string are

$$\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi} \cos\frac{\pi\sigma}{\sigma_1} \left(\cos\frac{\pi c t}{\sigma_1}, \sin\frac{\pi c t}{\sigma_1} \right). \tag{19.20}$$

The x, y components of the momentum density are

$$\vec{\mathcal{P}}^\tau(t,\sigma) = \frac{T_0}{c}\frac{\partial \vec{X}}{\partial t} = \frac{T_0}{c} \cos\frac{\pi\sigma}{\sigma_1} \left(-\sin\frac{\pi c t}{\sigma_1}, \cos\frac{\pi c t}{\sigma_1} \right). \tag{19.21}$$

The angular momentum (19.19) is then given by the integral

$$M_{12} = \frac{\sigma_1}{\pi}\frac{T_0}{c} \int_0^{\sigma_1} \cos^2\frac{\pi\sigma}{\sigma_1} d\sigma = \frac{\sigma_1^2 T_0}{2\pi c}. \tag{19.22}$$

Now the parametrization $d\sigma \propto dE$ implies that $\sigma_1 \propto E$, and in fact the energy of the string is $E = T_0\sigma_1$. Thus the angular momentum $J = |M_{12}|$ of a classical relativistic string is proportional to the square of its total energy

$$J = \frac{E^2}{2\pi T_0 c}. \tag{19.23}$$

This rule is obeyed by many meson and baryon resonances. The nucleon and five baryon resonances fit it with nearly the same value of the string tension

$$T_0 \approx 0.92 \ \ \text{GeV/fm} \tag{19.24}$$

as shown by Figs. 19.1, which displays the **Regge trajectories** of the N and Δ resonances on a single curve. Other N and Δ resonances, however, do not fall on this curve.