and the number of points 10^m rises further, the probability distributions $\Pr^{(m)}_{e,G,G}(-\infty, u)$ converge to the universal cumulative probability distribution $K(u)$ and provide a numerical verification of Kolmogorov's theorem. Such curves make poor figures, however, because they hide beneath $K(u)$. The curves labeled $\Pr^{(m)}_{e,S,G}(-\infty, u)$ for $m = 2$ and 3 are made from 100 sets of $N = 10^m$ points taken from $P_S(x, 3, 1)$ and tested as to whether they instead come from $P_G(x, 0, 1)$. Note that as $N = 10^m$ increases from 100 to 1000, the cumulative probability distribution $\Pr^{(m)}_{e,S,G}(-\infty, u)$ moves farther from Kolmogorov's universal cumulative probability distribution $K(u)$. In fact, the curve $\Pr^{(4)}_{e,S,G}(-\infty, u)$ made from 100 sets of 10^4 points lies beyond $u > 8$, too far to the right to fit in the figure. Kolmogorov's test gets more conclusive as the number of points $N \to \infty$. □

Warning, mathematical hazard: While binned data are ideal for chi-squared fits, they ruin Kolmogorov tests. The reason is that if the data are in bins of width w, then the empirical cumulative probability distribution $\Pr^{(N)}_e(-\infty, x)$ is a staircase function with steps as wide as the bin-width w even in the limit $N \to \infty$. Thus **even if the data come from the theoretical distribution**, the limiting value D_∞ of the Kolmogorov distance will be positive. In fact, one may show (exercise 13.43) that when the data do come from the theoretical probability distribution $P_t(x)$ assumed to be continuous, then the value of D_∞ is

$$D_\infty \approx \sup_{-\infty<x<\infty} \frac{w\, P_t(x)}{2}. \tag{13.324}$$

Thus in this case, the quantity $\sqrt{N}\, D_N$ would diverge as $\sqrt{N}\, D_\infty$ and lead one to believe that the data had not come from $P_t(x)$.

Suppose we have made some changes in our experimental apparatus and our software, and we want to see whether the new data $x'_1, x'_2, \ldots, x'_{N'}$ we took after the changes are consistent with the old data $x_1, x_2, \ldots, x_N$ we took before the changes. Then following equations (13.310–13.312), we can make two empirical cumulative probability distributions—one $\Pr^{(N)}_e(-\infty, x)$ made from the N old points x_j and the other $\Pr^{(N')}_e(-\infty, x)$ made from the N' new points x'_j. Next, we compute the distances

$$\begin{aligned} D^+_{N,N'} &= \sup_{-\infty<x<\infty} \Big(\Pr^{(N)}_e(-\infty, x) - \Pr^{(N')}_e(-\infty, x) \Big) \\ D_{N,N'} &= \sup_{-\infty<x<\infty} \Big| \Pr^{(N)}_e(-\infty, x) - \Pr^{(N')}_e(-\infty, x) \Big| \end{aligned} \tag{13.325}$$

which are analogous to (13.313–13.316). Smirnov (Smirnov 1939; Gnedenko