in which $G = 6.7087 \times 10^{-39}\, \hbar c\, (\mathrm{GeV}/c^2)^{-2} = 6.6742 \times 10^{-11}\, \mathrm{m}^3\, \mathrm{kg}^{-1}\, \mathrm{s}^{-2}$ is Newton's constant. Taking the trace and using $g^{ji}\, g_{ij} = \delta^j_j = 4$, we relate the scalar curvature to the trace $T = T^i{}_i$ of the energy-momentum tensor

$$R = \frac{8\pi G}{c^4}\, T. \tag{11.377}$$

So another form of Einstein's equations (11.376) is

$$R_{ij} = -\,\frac{8\pi G}{c^4}\left(T_{ij} - \frac{T}{2}\, g_{ij}\right). \tag{11.378}$$

On small scales, such as that of our solar system, one may neglect dark energy. So in empty space and on small scales, the energy-momentum tensor vanishes $T_{ij} = 0$ along with its trace and the scalar curvature $T = 0 = R$, and Einstein's equations are

$$R_{ij} = 0. \tag{11.379}$$

11.44 The Action of General Relativity

If we make an action that is a scalar, invariant under general coordinate transformations, and then apply to it the principle of stationary action, we will get tensor field equations that are invariant under general coordinate transformations. If the metric of space-time is among the fields of the action, then the resulting theory will be a possible theory of gravity. If we make the action as simple as possible, it will be Einstein's theory.

To make the action of the gravitational field, we need a scalar. Apart from the volume 4-form $*1 = \sqrt{g}\; d^4x = \sqrt{g}\; c\, dt\, d^3x$, the simplest scalar we can form from the metric tensor and its first and second derivatives is the scalar curvature R which gives us the **Einstein-Hilbert action**

$$S_{EH} = -\,\frac{c^3}{16\pi G}\int R\,\sqrt{g}\; d^4x = -\,\frac{c^3}{16\pi G}\int g^{ik}\, R_{ik}\,\sqrt{g}\; d^4x. \tag{11.380}$$

If $\delta g^{ik}(x)$ is a tiny change in the inverse metric, then we may write the first-order change in the action S_{EH} as (exercise 11.45)

$$\delta S_{EH} = -\,\frac{c^3}{16\pi G}\int \left(R_{ik} - \frac{1}{2}\, g_{ik} R\right)\sqrt{g}\; \delta g^{ik}\, d^4x. \tag{11.381}$$

Thus the principle of least action $\delta S_{EH} = 0$ leads to Einstein's equations

$$G_{ik} \equiv R_{ik} - \frac{1}{2}\, g_{ik}\, R = 0 \tag{11.382}$$

for empty space in which G_{ik} is Einstein's tensor.

The stress-energy tensor T_{ik} is defined so that the change in the action of the matter fields due to a tiny change $\delta g^{ik}(x)$ (vanishing at infinity) in the metric is

$$\delta S_m = -\frac{1}{2c}\int T_{ik}\sqrt{g}\;\delta g^{ik}\,d^4x. \tag{11.383}$$

So the principle of least action $\delta S = \delta S_{EH} + \delta S_m = 0$ implies Einstein's equations (11.376, 11.378) in the presence of matter and energy

$$R_{ik} - \frac{1}{2}\,g_{ik}\,R = -\frac{8\pi G}{c^4}T_{ij} \quad \text{or} \quad R_{ij} = -\frac{8\pi G}{c^4}\left(T_{ij} - \frac{T}{2}g_{ij}\right). \tag{11.384}$$

11.45 Standard Form

Tensor equations are independent of the choice of coordinates, so it's wise to choose coordinates that simplify one's work. For a **static** and **isotropic** gravitational field, this choice is the **standard form** (Weinberg, 1972, ch. 8)

$$d\tau^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \tag{11.385}$$

in which $c = 1$, and $B(r)$ and $A(r)$ are functions that one may find by solving the field equations (11.376). Since $d\tau^2 = -\,ds^2 = -g_{ij}\,dx^i dx^j$, the nonzero components of the metric tensor are $g_{rr} = A(r)$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2\sin^2\theta$, and $g_{00} = -B(r)$, and those of its inverse are $g^{rr} = A^{-1}(r)$, $g^{\theta\theta} = r^{-2}$, $g^{\phi\phi} = r^{-2}\sin^{-2}\theta$, and $g^{00} = -B^{-1}(r)$. By differentiating the metric tensor and using (11.255), one gets the components of the connection $\Gamma^i_{k\ell}$, such as $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, and the components (11.353) of the Ricci tensor R_{ij}, such as (Weinberg, 1972, ch. 8)

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) \tag{11.386}$$

in which the primes mean d/dr.

11.46 Schwarzschild's Solution

If one ignores the small dark-energy parameter Λ, one may solve Einstein's field equations (11.379) in empty space

$$R_{ij} = 0 \tag{11.387}$$