How weak are the static gravitational fields we know about? The dimensionless ratio ϕ/c^2 is 10^{-39} on the surface of a proton, 10^{-9} on the Earth, 10^{-6} on the surface of the sun, and 10^{-4} on the surface of a white dwarf.

11.41 Gravitational Time Dilation

Suppose we have a system of coordinates x^i with a metric g_{ik} and a clock at rest in this system. Then the proper time $d\tau$ between ticks of the clock is

$$d\tau = (1/c)\sqrt{-g_{ij}\,dx^i\,dx^j} = \sqrt{-g_{00}}\,dt \tag{11.340}$$

where dt is the time between ticks in the x^i coordinates, which is the laboratory frame in the gravitational field g_{00}. By the principle of equivalence (section 11.39), the proper time $d\tau$ between ticks is the same as the time between ticks when the same clock is at rest deep in empty space.

If the clock is in a weak static gravitational field due to a mass M at a distance r, then

$$-\,g_{00} = 1 + 2\phi/c^2 = 1 - 2GM/c^2r \tag{11.341}$$

is a little less than unity, and the interval of proper time between ticks

$$d\tau = \sqrt{-g_{00}}\,dt = \sqrt{1 - 2GM/c^2r}\,dt \tag{11.342}$$

is slightly less than the interval dt between ticks in the coordinate system of an observer at x in the rest frame of the clock and the mass, and in its gravitational field. Since $dt > d\tau$, the laboratory time dt between ticks is greater than the proper or intrinsic time $d\tau$ between ticks of the clock unaffected by any gravitational field. Clocks near big masses run slow.

Now suppose we have two identical clocks in at different heights above sea level. The time T_ℓ for the lower clock to make N ticks will be longer than the time T_u for the upper clock to make N ticks. The ratio of the clock times will be

$$\frac{T_\ell}{T_u} = \frac{\sqrt{1 - 2GM/c^2(r+h)}}{\sqrt{1 - 2GM/c^2r}} \approx 1 + \frac{gh}{c^2}. \tag{11.343}$$

Now imagine that a photon going down passes the upper clock which measures its frequency as ν_u and then passes the lower clock which measures its frequency as ν_ℓ. The slower clock will measure a higher frequency. The ratio of the two frequencies will be the same as the ratio of the clock times

$$\frac{\nu_\ell}{\nu_u} = 1 + \frac{gh}{c^2}. \tag{11.344}$$

As measured by the lower, slower clock, the photon is blue shifted.