is the 2-form

$$\begin{aligned} dA &= \partial_y A_x \, dy \wedge dx + \partial_z A_x \, dz \wedge dx \\ &\quad + \partial_x A_y \, dx \wedge dy + \partial_z A_y \, dz \wedge dy \\ &\quad + \partial_x A_z \, dx \wedge dz + \partial_y A_z \, dy \wedge dz \\ &= (\partial_y A_z - \partial_z A_y) \, dy \wedge dz \\ &\quad + (\partial_z A_x - \partial_x A_z) \, dz \wedge dx \\ &\quad + (\partial_x A_y - \partial_y A_x) \, dx \wedge dy \\ &= (\nabla \times A)_x \, dy \wedge dz + (\nabla \times A)_y \, dz \wedge dx + (\nabla \times A)_z \, dx \wedge dy \end{aligned} \tag{11.111}$$

in which we recognize the curl (6.39) of $\boldsymbol{A}$. □

The exterior derivative of the 1-form $A = A_j \, dx^j$ is the 2-form

$$dA = dA_j \wedge dx^j = \partial_i A_j \, dx^i \wedge dx^j = \tfrac{1}{2} F_{ij} \, dx^i \wedge dx^j = F \tag{11.112}$$

in which $\partial_i = \partial/\partial x^i$. So d turns the electromagnetic 1-form A—the 4-vector potential or gauge field A_j—into the Faraday 2-form—the tensor F_{ij}. Its square dd vanishes: dd applied to any p-form Q is zero

$$ddQ_{i\ldots} dx^i \wedge \cdots = d(\partial_r Q_{i\ldots}) \wedge dx^r \wedge dx^i \wedge \cdots = (\partial_s \partial_r Q_{i\ldots}) dx^s \wedge dx^r \wedge dx^i \wedge \cdots = 0 \tag{11.113}$$

because $\partial_s \partial_r Q$ is symmetric in r and s while $dx^s \wedge dx^r$ is anti-symmetric.

Some writers drop the wedges and write $dx^i \wedge dx^j$ as $dx^i dx^j$ while keeping the rules of antisymmetry $dx^i dx^j = -dx^j dx^i$ and $\left(dx^i\right)^2 = 0$. But this economy prevents one from using invariant quantities like $S = \frac{1}{2} S_{ik} \, dx^i dx^k$ in which S_{ik} is a second-rank covariant symmetric tensor. If M_{ik} is a covariant second-rank tensor with no particular symmetry, then (exercise 11.7) only its antisymmetric part contributes to the 2-form $M_{ik} \, dx^i \wedge dx^k$ and only its symmetric part contributes to the quantity $M_{ik} \, dx^i dx^k$.

The exterior derivative d applied to the Faraday 2-form $F = dA$ gives

$$dF = ddA = 0 \tag{11.114}$$

which is the Bianchi identity (11.93). A p-form H is **closed** if $dH = 0$. By (11.114), the Faraday 2-form is closed, $dF = 0$.

A p-form H is **exact** if there is a $(p-1)$-form K whose differential is $H = dK$. The identity (11.113) or $dd = 0$ implies that **every exact form is closed**. The lemma of Poincaré shows that **every closed form is locally exact**.

If the A_i in the 1-form $A = A_i dx^i$ commute with each other, then the 2-form $A \wedge A = 0$. But if the A_i don't commute because they are matrices or operators or Grassmann variables, then $A \wedge A$ need not vanish.

Example 11.12 (A Static Electric Field Is Closed and Locally Exact) If $\dot{\boldsymbol{B}} = 0$, then by Faraday's law (11.82) the curl of the electric field vanishes, $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$. Writing the electrostatic field as the 1-form $E = E_i \, dx^i$ for $i = 1, 2, 3$, we may express the vanishing of its curl as

$$dE = \partial_j E_i \, dx^j \wedge dx^i = \frac{1}{2} \left(\partial_j E_i - \partial_i E_j \right) \, dx^j \wedge dx^i = 0 \tag{11.115}$$

which says that E is closed. We can define a quantity $V_P(\boldsymbol{x})$ as a line integral of the 1-form E along a path P to $\boldsymbol{x}$ from some starting point $\boldsymbol{x}_0$

$$V_P(\boldsymbol{x}) = -\int_{P, \boldsymbol{x}_0}^{\boldsymbol{x}} E_i \, dx^i = -\int_P E \tag{11.116}$$

and so $V_P(\boldsymbol{x})$ will depend on the path P as well as on $\boldsymbol{x}_0$ and $\boldsymbol{x}$. But if $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ in some ball (or neighborhood) around $\boldsymbol{x}$ and $\boldsymbol{x}_0$, then within that ball the dependence on the path P drops out because the difference $V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x})$ is the line integral of E around a closed loop in the ball which by Stokes's theorem (6.44) is an integral of the vanishing curl $\boldsymbol{\nabla} \times \boldsymbol{E}$ over any surface S in the ball whose boundary ∂S is the closed curve $P' - P$

$$V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x}) = \oint_{P'-P} E_i \, dx^i = \int_S (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot da = 0 \tag{11.117}$$

or

$$V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x}) = \int_{\partial S} E = \int_S dE = 0 \tag{11.118}$$

in the language of forms (George Stokes, 1819–1903). Thus the potential $V_P(\boldsymbol{x}) = V(\boldsymbol{x})$ is independent of the path, $\boldsymbol{E} = -\boldsymbol{\nabla} V(\boldsymbol{x})$, and the 1-form $E = E_i \, dx^i = -\partial_i V \, dx^i = -dV$ is locally exact. □

The general form of Stokes's theorem is that the integral of any p-form H over the boundary ∂R of any $(p+1)$-dimensional, simply connected, orientable region R is equal to the integral of the $(p+1)$-form dH over R

$$\int_{\partial R} H = \int_R dH \tag{11.119}$$

which for $p = 1$ gives (6.44).

Example 11.13 (Stokes's Theorem for 0-forms) Here $p = 0$, the region $R = [a, b]$ is 1-dimensional, H is a 0-form, and Stokes's theorem is

$$H(b) - H(a) = \int_{\partial R} H = \int_R dH = \int_a^b dH(x) = \int_a^b H'(x) \, dx \tag{11.120}$$

familiar from elementary calculus. □