The metric η raises (11.31) and lowers (11.33) the index of a basis vector.

The component x'^i is related to the components x^j by the linear map

$$x'^i = e'^i \cdot p = e'^i \cdot e_j \, x^j. \tag{11.34}$$

Such a map from a 4-vector x to a 4-vector x' is a **Lorentz transformation**

$$x'^i = L^i_j \, x^j \quad \text{with matrix} \quad L^i_j = e'^i \cdot e_j. \tag{11.35}$$

The inner product (p, q) of two points $p = e_i \, x^i = e'_i \, x'^i$ and $q = e_k \, y^k = e'_k \, y'^k$ is **physical** and so is **invariant under Lorentz transformations**

$$(p, q) = x^i \, y^k \, e_i \cdot e_k = \eta_{ik} \, x^i \, y^k = x'^i \, y'^k \, e'_i \cdot e'_k = \eta_{ik} \, x'^i \, y'^k. \tag{11.36}$$

With $x'^i = L^i{}_r \, x^r$ and $y'^k = L^k{}_s \, y^s$, this invariance is

$$\eta_{rs} \, x^r y^s = \eta_{ik} \, L^i{}_r \, x^r \, L^k{}_s \, y^s = \eta_{ik} \, x'^i y'^k \tag{11.37}$$

or since x^r and y^s are arbitrary

$$\eta_{rs} = \eta_{ik} \, L^i{}_r \, L^k{}_s = L^i{}_r \, \eta_{ik} \, L^k{}_s. \tag{11.38}$$

In matrix notation, a left index labels a row, and a right index labels a column. Transposition interchanges rows and columns $L^i{}_r = L^{\mathsf{T}}{}_r{}^i$, so

$$\eta_{rs} = L^{\mathsf{T}}{}_r{}^i \, \eta_{ik} \, L^k{}_s \quad \text{or} \quad \eta = L^{\mathsf{T}} \, \eta \, L \tag{11.39}$$

in matrix notation. In such matrix products, the height of an index—whether it is up or down—determines whether it is contravariant or covariant but does not affect its place in its matrix.

Example 11.4 (A Boost) The matrix

$$L = \begin{pmatrix} \gamma & \sqrt{\gamma^2 - 1} & 0 & 0 \\ \sqrt{\gamma^2 - 1} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.40}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ represents a Lorentz transformation that is a boost in the x-direction. Boosts and rotations are Lorentz transformations. Working with 4×4 matrices can get tedious, so students are advised to think in terms of scalars, like $p \cdot x = p^i \eta_{ij} x^j = \boldsymbol{p} \cdot \boldsymbol{x} - Et$ whenever possible. □

If the basis vectors e and e' are independent of p and of x, then the coefficients of the transformation law (11.6) for contravariant vectors are

$$\frac{\partial x'^i}{\partial x^j} = e'^i \cdot e_j. \tag{11.41}$$