and generate the elements of the group $SU(2)$

$$\exp\left(i\,\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right) = I\,\cos\frac{\theta}{2} + i\,\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\,\sin\frac{\theta}{2} \tag{10.118}$$

in which I is the 2×2 identity matrix, $\theta = \sqrt{\boldsymbol{\theta}^2}$ and $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}/\theta$.

It follows from (10.117) that the spin operators satisfy

$$[S_a, S_b] = i\,\hbar\,\epsilon_{abc}\,S_c. \tag{10.119}$$

The raising and lowering operators

$$S_\pm = S_1 \pm iS_2 \tag{10.120}$$

have simple commutators with S_3

$$[S_3, S_\pm] = \pm\,\hbar\,S_\pm. \tag{10.121}$$

This relation implies that if the state $|j,m\rangle$ is an eigenstate of S_3 with eigenvalue $\hbar m$, then the states $S_\pm|j,m\rangle$ either vanish or are eigenstates of S_3 with eigenvalues $\hbar(m\pm1)$

$$S_3S_\pm|j,m\rangle = S_\pm S_3|j,m\rangle \pm \hbar S_\pm|j,m\rangle = \hbar(m\pm1)S_\pm|j,m\rangle. \tag{10.122}$$

Thus the raising and lowering operators raise and lower the eigenvalues of S_3. When $j = 1/2$, the possible values of m are $m = \pm1/2$, and so with the usual sign and normalization conventions

$$S_+|-\rangle = \hbar|+\rangle \quad \text{and} \quad S_-|+\rangle = \hbar|-\rangle \tag{10.123}$$

while

$$S_+|+\rangle = 0 \quad \text{and} \quad S_-|-\rangle = 0. \tag{10.124}$$

The square of the total spin operator is simply related to the raising and lowering operators and to S_3

$$\boldsymbol{S}^2 = S_1^2 + S_2^2 + S_3^2 = \frac{1}{2}S_+S_- + \frac{1}{2}S_-S_+ + S_3^2. \tag{10.125}$$

But the squares of the Pauli matrices are unity, and so $S_a^2 = (\hbar/2)^2$ for all three values of a. Thus

$$\boldsymbol{S}^2 = \frac{3}{4}\,\hbar^2 \tag{10.126}$$

is a Casimir operator (10.105) for a spin one-half system.

Example 10.20 (Two Spin 1/2's) Consider two spin operators $\boldsymbol{S}^{(1)}$ and