Thus $U(g)$ cannot change $E$ or $j$, and so

$$\langle E', j', m'|U(g)|E, j, m\rangle = \delta_{E'E}\delta_{j'j}\langle j, m'|U(g)| j, m\rangle = \delta_{E'E}\delta_{j'j}D^{(j)}_{m'm}(g). \tag{10.13}$$

The matrix element (10.11) is a single sum over $E$ and $j$ in which the irreducible representations $D^{(j)}_{m'm}(g)$ of the rotation group $SU(2)$ appear

$$\langle\phi|U(g)|\psi\rangle = \sum_{E,j,m',m} \langle\phi|E, j, m'\rangle D^{(j)}_{m'm}(g)\langle E, j, m|\psi\rangle. \tag{10.14}$$

This is how the block-diagonal form (10.7) usually appears in calculations. The matrices $D^{(j)}_{m'm}(g)$ inherit the unitarity of the operator $U(g)$. □

## 10.4 Subgroups

If all the elements of a group $S$ also are elements of a group $G$, then $S$ is a **subgroup** of $G$. Every group $G$ has two **trivial subgroups**—the identity element $e$ and the whole group $G$ itself. Many groups have more interesting subgroups. For example, the rotations about a fixed axis is an abelian subgroup of the group of all rotations in 3-dimensional space.

A subgroup $S \subset G$ is an **invariant** subgroup if every element $s$ of the subgroup $S$ is left inside the subgroup under the **action** of every element $g$ of the whole group $G$, that is, if

$$g^{-1} s\, g = s' \in S \quad \text{for all} \quad g \in G. \tag{10.15}$$

This condition often is written as $g^{-1}Sg = S$ for all $g \in G$ or as

$$S\, g = g\, S \quad \text{for all } g \in G. \tag{10.16}$$

Invariant subgroups also are called **normal subgroups**.

A set $C \subset G$ is called a **conjugacy class** if it's invariant under the action of the whole group $G$, that is, if $Cg = g\, C$ or

$$g^{-1} C\, g = C \quad \text{for all } g \in G. \tag{10.17}$$

A subgroup that is the union of a set of conjugacy classes is invariant.

The **center** $C$ of a group $G$ is the set of all elements $c \in G$ that **commute** with every element $g$ of the group, that is, their **commutators**

$$[c, g] \equiv cg - gc = 0 \tag{10.18}$$

vanish for all $g \in G$.

**Example 10.8** (Centers Are Abelian Subgroups) Does the center $C$ always form an abelian subgroup of its group $G$? The product $c_1 c_2$ of any two elements $c_1$ and $c_2$ of the center commutes with every element $g$ of $G$ since $c_1 c_2 g = c_1 g c_2 = g c_1 c_2$. So the center is closed under multiplication. The identity element $e$ commutes with every $g \in G$, so $e \in C$. If $c' \in C$, then $c' g = g c'$ for all $g \in G$, and so multiplication of this equation from the left and the right by $c'^{-1}$ gives $g c'^{-1} = c'^{-1} g$, which shows that $c'^{-1} \in C$. The subgroup $C$ is abelian because each of its elements commutes with all the elements of G including those of $C$ itself. □

So the center of any group always is one of its abelian invariant subgroups. The center may be trivial, however, consisting either of the identity or of the whole group. But a group with a nontrivial center can not be simple or semisimple (section 10.23).

## 10.5 Cosets

If $H$ is a subgroup of a group $G$, then for every element $g \in G$ the set of elements $Hg \equiv \{hg | h \in H, g \in G\}$ is a **right coset of the subgroup $\boldsymbol{H \subset G}$**. (Here $\subset$ means *is a subset of* or equivalently *is contained in*.)

If $H$ is a subgroup of a group $G$, then for every element $g \in G$ the set of elements $gH$ is a **left coset of the subgroup $\boldsymbol{H \subset G}$**.

The number of elements in a coset is the same as the number of elements of $H$, which is the order of $H$.

An element $g$ of a group $G$ is in one and only one right coset (and in one and only one left coset) of the subgroup $H \subset G$. For suppose instead that $g$ were in two right cosets $g \in Hg_1$ and $g \in Hg_2$, so that $g = h_1 g_1 = h_2 g_2$ for suitable $h_1, h_2 \in H$ and $g_1, g_2 \in G$. Then since $H$ is a (sub)group, we have $g_2 = h_2^{-1} h_1 g_1 = h_3 g_1$, which says that $g_2 \in Hg_1$. But this means that every element $hg_2 \in Hg_2$ is of the form $hg_2 = hh_3 g_1 = h_4 g_1 \in Hg_1$. So every element $hg_2 \in Hg_2$ is in $Hg_1$: the two right cosets are identical, $Hg_1 = Hg_2$.

The right (or left) cosets are the points of the **quotient coset space** $G/H$.

If $H$ is an invariant subgroup of $G$, then by definition (10.16) $Hg = gH$ for all $g \in G$, and so the left cosets are the same sets as the right cosets. In this case, the coset space $G/H$ is itself a group with multiplication defined