$\sin x/x$ and $j_1(x) = \sin x/x^2 - \cos x/x$. Rayleigh's formula leads to the recursion relation (exercise 9.22)

$$j_{\ell+1}(x) = \frac{\ell}{x} j_\ell(x) - j'_\ell(x) \tag{9.69}$$

with which one can show (exercise 9.23) that the spherical Bessel functions as defined by Rayleigh's formula do satisfy their differential equation (9.66) with $x = kr$.

The spherical Bessel functions $j_\ell(kr)$ satisfy the self-adjoint Sturm-Liouville (6.333) equation (9.66)

$$-r^2 j''_\ell - 2r j'_\ell + \ell(\ell+1) j_\ell = k^2 r^2 j_\ell \tag{9.70}$$

with eigenvalue k^2 and weight function $\rho = r^2$. If $j_\ell(z_{\ell,n}) = 0$, then the functions $j_\ell(kr) = j_\ell(z_{\ell,n} r/a)$ vanish at $r = a$ and form an orthogonal basis

$$\int_0^a j_\ell(z_{\ell,n} r/a)\, j_\ell(z_{\ell,m} r/a)\, r^2\, dr = \frac{a^3}{2}\, j^2_{\ell+1}(z_{\ell,n})\, \delta_{n,m} \tag{9.71}$$

for a self-adjoint system on the interval $[0, a]$. Moreover, since as $n \to \infty$ the eigenvalues $k^2_{\ell,n} = z^2_{\ell,n}/a^2 \approx [(n + \ell/2)\pi]^2/a^2 \to \infty$, the eigenfunctions $j_\ell(z_{\ell,n} r/a)$ also are complete in the mean (section 6.35).

On an infinite interval, the analogous relation is

$$\int_0^\infty j_\ell(kr)\, j_\ell(k'r)\, r^2\, dr = \frac{\pi}{2k^2}\, \delta(k - k'). \tag{9.72}$$

If we write the spherical Bessel function $j_0(x)$ as the integral

$$j_0(z) = \frac{\sin z}{z} = \frac{1}{2} \int_{-1}^{1} e^{izx}\, dx \tag{9.73}$$

and use Rayleigh's formula (9.68), we may find an integral for $j_\ell(z)$

$$\begin{aligned} j_\ell(z) &= (-1)^\ell z^\ell \left(\frac{1}{z}\frac{d}{dz}\right)^\ell \left(\frac{\sin z}{z}\right) = (-1)^\ell z^\ell \left(\frac{1}{z}\frac{d}{dz}\right)^\ell \frac{1}{2}\int_{-1}^{1} e^{izx}\, dx \\ &= \frac{z^\ell}{2}\int_{-1}^{1} \frac{(1-x^2)^\ell}{2^\ell \ell!}\, e^{izx}\, dx = \frac{(-i)^\ell}{2}\int_{-1}^{1} \frac{(1-x^2)^\ell}{2^\ell \ell!}\, \frac{d^\ell}{dx^\ell} e^{izx}\, dx \\ &= \frac{(-i)^\ell}{2}\int_{-1}^{1} e^{izx}\, \frac{d^\ell}{dx^\ell}\frac{(x^2-1)^\ell}{2^\ell \ell!}\, dx = \frac{(-i)^\ell}{2}\int_{-1}^{1} P_\ell(x)\, e^{izx}\, dx \end{aligned} \tag{9.74}$$

(exercise 9.24) that contains Rodrigues's formula (8.8) for the Legendre polynomial $P_\ell(x)$. With $z = kr$ and $x = \cos\theta$, this formula

$$i^\ell\, j_\ell(kr) = \frac{1}{2}\int_{-1}^{1} P_\ell(\cos\theta) e^{ikr\cos\theta}\, d\cos\theta \tag{9.75}$$