at $\rho = r$ as well as the separable wave equations (9.57). The frequencies of the resonant TE modes then are $\omega_{n,m,\ell} = c\sqrt{z'^2_{n,m}/r^2 + \pi^2\ell^2/h^2}$.

The TM modes are $B_z = 0$ and

$$E_z = J_n(z_{n,m}\,\rho/r)\,e^{in\phi}\,\sin(\pi\ell z/h)\,e^{-i\omega t} \tag{9.62}$$

with resonant frequencies $\omega_{n,m,\ell} = c\sqrt{z^2_{n,m}/r^2 + \pi^2\ell^2/h^2}$. □

## 9.2 Spherical Bessel functions of the first kind

If in Bessel's equation (9.4), one sets $n = \ell + 1/2$ and $j_\ell = \sqrt{\pi/2x}\,J_{\ell+1/2}$, then one may show (exercise 9.21) that

$$x^2\,j''_\ell(x) + 2x\,j'_\ell(x) + [x^2 - \ell(\ell+1)]\,j_\ell(x) = 0 \tag{9.63}$$

which is the equation for the **spherical Bessel function** $j_\ell$.

We saw in example 6.6 that by setting $V(r,\theta,\phi) = R_{k,\ell}(r)\,\Theta_{\ell,m}(\theta)\,\Phi_m(\phi)$ we could separate the variables of Helmholtz's equation $-\triangle V = k^2 V$ in spherical coordinates

$$\frac{r^2\triangle V}{V} = \frac{(r^2 R'_{k,\ell})'}{R_{k,\ell}} + \frac{(\sin\theta\,\Theta'_{\ell,m})'}{\sin\theta\,\Theta_{\ell,m}} + \frac{\Phi''}{\sin^2\theta\,\Phi} = -k^2 r^2. \tag{9.64}$$

Thus if $\Phi_m(\phi) = e^{im\phi}$ so that $\Phi''_m = -m^2\Phi_m$, and if $\Theta_{\ell,m}$ satisfies the **associated Legendre equation** (8.91)

$$\sin\theta\left(\sin\theta\,\Theta'_{\ell,m}\right)' + [\ell(\ell+1)\sin^2\theta - m^2]\,\Theta_{\ell,m} = 0 \tag{9.65}$$

then the product $V(r,\theta,\phi) = R_{k,\ell}(r)\,\Theta_{\ell,m}(\theta)\,\Phi_m(\phi)$ will obey (9.64) because in view of (9.63) the radial function $R_{k,\ell}(r) = j_\ell(kr)$ satisfies

$$(r^2 R'_{k,\ell})' + [k^2 r^2 - \ell(\ell+1)]R_{k,\ell} = 0. \tag{9.66}$$

In terms of the spherical harmonic $Y_{\ell,m}(\theta,\phi) = \Theta_{\ell,m}(\theta)\,\Phi_m(\phi)$, the solution is $V(r,\theta,\phi) = j_\ell(kr)\,Y_{\ell,m}(\theta,\phi)$.

Rayleigh's formula gives the spherical Bessel function

$$j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}}\,J_{\ell+1/2}(x) \tag{9.67}$$

as the $\ell$th derivative of $\sin x/x$

$$j_\ell(x) = (-1)^\ell\,x^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\left(\frac{\sin x}{x}\right) \tag{9.68}$$