solutions $\alpha \cos x + \beta \sin x$ of the homogeneous equation $-f'' - f = 0$, the functions $u(x) = \sin x$ and $v(x) = \cos x$. Substituting them into the formula (6.413) and setting $p(x) = 1$ and $A = -W(x_0) = \sin^2(x_0) + \cos^2(x_0) = 1$, we find as the Green's function

$$G(x, y) = \theta(x - y) \sin y \cos x + \theta(y - x) \sin x \cos y. \tag{6.419}$$

The solution then is

$$\begin{aligned} f(x) &= \int_a^b G(x, y)\, e^y\, dy \\ &= \int_{-\pi}^{\pi} [\theta(x - y) \sin y \cos x + \theta(y - x) \sin x \cos y]\, e^y\, dy \\ &= \cos x \int_{-\pi}^{x} e^y\, \sin y\, dy + \sin x \int_x^{\pi} e^y\, \cos y\, dy \\ &= -\frac{1}{2} \left(e^{-\pi} \cos x + e^{\pi} \sin x + e^x \right). \end{aligned} \tag{6.420}$$

□

6.40 Nonlinear Differential Equations

The field of nonlinear differential equations is too vast to cover here, but we may hint at some of its features by considering some examples from cosmology and particle physics.

The Friedmann equations of general relativity (11.413 & 11.415) for the dimensionless scale factor $a(t)$ of a homogeneous, isotropic universe are (in natural units, $c = 1$)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p) \quad \text{and} \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \tag{6.421}$$

in which k respectively is 1, 0, and -1 for closed, flat, and open geometries. (The scale factor $a(t)$ tells how much space has expanded or contracted by the time t.) These equations become more tractable when the energy density ρ is due to a single constituent whose pressure p is related to it by an equation of state $p = w\rho$. Conservation of energy $\dot{\rho} = -3\,\dot{a}(\rho + p)/a$ (11.429–11.434) then ensures (exercise 6.30) that the product $\rho\, a^{3(1+w)}$ is independent of time. The constant w respectively is $1/3$, 0, and -1 for